

LOCAL WELL-POSEDNESS AND BREAK-DOWN CRITERION OF THE INCOMPRESSIBLE EULER EQUATIONS WITH FREE BOUNDARY

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ABSTRACT. In this paper, we prove the local well-posedness of the free boundary problem for the incompressible Euler equations in low regularity Sobolev spaces, in which the velocity is a Lipschitz function and the free surface belongs to $C^{\frac{3}{2}+\varepsilon}$. Moreover, we also present a Beale-Kato-Majda type break-down criterion of smooth solution in terms of the mean curvature of the free surface, the gradient of the velocity and Taylor sign condition.

1. INTRODUCTION

1.1. Presentation of the problem. In this paper, we consider the motion of an ideal incompressible gravity fluid in a domain with free boundary of finite depth

$$\{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega_t\},$$

where Ω_t is the fluid domain at time t located by

$$\Omega_t = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : b(x) < y < \eta(t, x)\}.$$

The motion of the fluid is described by the incompressible Euler equation

$$\partial_t v + v \cdot \nabla_{x,y} v = -ge_{d+1} - \nabla_{x,y} P \quad \text{in } \Omega_t, \quad t \geq 0, \quad (1.1)$$

where $-ge_{d+1} = -g(0, \dots, 0, 1)$ denotes the acceleration of gravity, $v = (v^1, \dots, v^{d+1})$ denotes the velocity field, and P denotes the pressure. The incompressibility of the fluid is expressed by

$$\operatorname{div} v = 0 \quad \text{in } \Omega_t, \quad t \geq 0. \quad (1.2)$$

Assume that no fluid particles are transported across the surface. At the bottom, this is given by

$$v_n|_{y=b(X)} := \mathbf{n}_- \cdot v|_{y=b(X)} = 0 \quad \text{for } t \geq 0, x \in \mathbf{R}^d \quad (1.3)$$

where $\mathbf{n}_- := \frac{1}{\sqrt{1+|\nabla_X b|^2}}(\nabla_X b, -1)^T$ denotes the outward normal vector to the lower boundary of Ω_t . At the free surface, the boundary condition is kinematic and is given by

$$\partial_t \eta - \sqrt{1+|\nabla \eta|^2} v_n|_{y=\eta(t,x)} = 0 \quad \text{for } t \geq 0, x \in \mathbf{R}^d, \quad (1.4)$$

where $v_n = \mathbf{n}_+ \cdot v|_{y=\eta(t,x)}$, with $\mathbf{n}_+ := \frac{1}{\sqrt{1+|\nabla \eta|^2}}(-\nabla \eta, 1)^T$ denoting the outward normal vector to the free surface Σ_t . In general, the pressure at the free surface is proportional to the mean curvature of the free surface, i.e.,

$$P|_{y=\eta(t,x)} = -\kappa \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}} \right) \quad \text{for } t \geq 0, x \in \mathbf{R}^d, \quad (1.5)$$

where $\kappa \geq 0$ is the surface tension coefficient.

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In this paper, we consider the case without surface tension. Furthermore, we assume that the bottom is flat (i.e., $b(x) = -1$) in order to simplify our presentation. We also take the gravity constant $g = 1$.

1.2. Some known results. Let us first review some known results concerning the water-wave equations without vorticity. In the case when the surface tension is neglected and the motion of free surface is a small perturbation of still water, one could check Nalimov [28], Yosihara [38] and Craig [17] for the well-posedness of 2-D water-wave equations. In general, the local well-posedness of the water-wave equations of infinite depth without surface tension was solved by Wu [33, 34], where she showed that the Taylor sign condition

$$-\frac{\partial p}{\partial \mathbf{n}}|_{y=\eta(t,x)} \geq c_0 > 0 \quad (1.6)$$

always holds as long as the free surface is no self-intersection. In [6, 7], Ambrose and Masmoudi present a different proof. Lannes [23] first solves the water-wave equations of finite depth without surface tension in the framework of the Eulerian coordinates. Ming and Zhang [27] generalize Lannes's result to the case with surface tension. In a series of works [1, 2, 3], Alazard, Burq and Zuily use the tools of paradifferential operators and Strichartz estimates to prove the local well-posedness of the water-wave equations in low regularity Sobolev spaces.

For small initial data, Wu [35] first proved the almost global well-posedness of 2-D water-wave equations. Wu [36] and Germain, Masmoudi and Shatah [19] proved the global well-posedness of 3-D water-wave equations by using different method. Recently, Alazard and Delort [5] and Ionescu and Pusateri [25] independently proved the global well-posedness of 2-D water-wave equations, see also [21, 22] for a new proof based on the holomorphic coordinates. On the other hand, Castro, Cordoba, Ferferman, Gancedo and Lopez-Fernandez [11] showed that there exists smooth initial data for the water-waves equations such that the solution overturns in finite time. See [12, 16] for the formation of the splash singularity. Wu [37] also construct a class of self-similar solution for the 2-D water-wave equations without the gravity.

Now, we review some well-posedness results for the rotational water-wave equations. Christodoulou and Lindblad [14] presented the a priori estimates of the incompressible Euler equations in a free domain diffeomorphic to a ball. Later, Lindblad [26] proved the local existence of smooth solution by using Nash-Moser iteration. Coutand and Shkoller [15] proved the local well-posedness of the incompressible Euler equations in both cases with surface tension and without surface tension by using the lagrangian coordinates and a subtle mollification procedure. Zhang and Zhang [39] solves the incompressible Euler equations without surface tension by using the framework of Clifford analysis introduced by Wu [34]. Shatah and Zeng [29, 30, 31] solve this problem by deriving the evolution equations of geometry quantities, especially the mean curvature.

In this paper, we will first prove the local well-posedness of the rotational water-wave problem in low regularity Sobolev spaces, and then present a break-down criterion to the obtained smooth solution in terms of physical quantity and geometrical quantity. This work was motivated by Craig and Wayne's question proposed in [18]: **"How do solutions break down?"**

There are several versions of this question, including "What is the lowest exponent of a Sobolev space H^s in which one can produce an existence theorem local in

time?" Or one could ask "For which α is it true that, if one knows a priori that $\sup_{[-T,T]} \|(\eta, \psi)\|_{C^\alpha} < +\infty$ and that $(\eta_0, \psi_0) \in C^\infty$, then the solution is fact C^∞ over the time interval $[-T, T]$?" It would be more satisfying to say that the solution fails to exist because the curvature of the surface has diverged at some point, or a related geometrical and(or) physical statement.

In the case without the vorticity and surface tension, this question was solved by Alazard, Burq and Zuily for the low regularity well-posedness [3], and by Wang and Zhang for the break-down criterion [32].

1.3. Main results. The first main result of this paper is the local well-posedness of the water-wave equations in Sobolev spaces with low regularity, where the regularity of the initial velocity is consistent with the classical local well-posedness result in \mathbf{R}^{d+1} proved by Kato and Ponce [20].

Theorem 1.1. *Let $d \geq 1$ and $s > \frac{d}{2} + 1$. Assume that the initial data (η_0, v_0) satisfies*

$$\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad v_0 \in H^{s+\frac{1}{2}}(\Omega_0).$$

Furthermore, assume that there exist two positive constants $c_0 > 0$ and $h_0 > 0$ such that

$$-(\partial_y P)(0, x, \eta_0(x)) \geq c_0 \quad \text{for } x \in \mathbf{R}^d, \quad (1.7)$$

$$1 + \eta_0(x) \geq h_0 \quad \text{for } x \in \mathbf{R}^d. \quad (1.8)$$

Then there exists $T > 0$ such that the system (1.1)–(1.5) with the initial data (η_0, v_0) has a unique solution (η, v) satisfying

$$\eta \in C([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d)), \quad v \in C([0, T]; H^{s+\frac{1}{2}}(\Omega_t)).$$

Remark 1.2. *The regularity of the initial velocity should be optimal by the recent strong ill-posedness result in the whole space proved by Bourgain and Li [10]. The regularity of the initial free surface could be further lowered by using the Strichartz type estimates, see [4] for the irrotational case.*

In a seminal paper [9], Beale, Kato and Majda showed that if v is a smooth solution of the incompressible Euler equations in $[0, T] \times \mathbf{R}^3$ and satisfies

$$\int_0^T \|\nabla \times v(t)\|_{L^\infty(\mathbf{R}^3)} dt < +\infty,$$

then the solution can be extended after $t = T$. The second main result of this paper is a Beale-Kato-Majda type blow-up criterion for the free boundary problem of the incompressible Euler equations.

Theorem 1.3. *Let $s > \frac{d}{2} + 1$ so that $s - \frac{1}{2}$ is an integer, and (η, v, P) be the solution of the system (1.1)–(1.5) in $[0, T]$ obtained in Theorem 1.1. If the solution (η, v, P) satisfies*

$$M(T) \stackrel{\text{def}}{=} \sup_{t \in [0, T]} (\|H(t)\|_{L^p \cap L^2} + \|v(t)\|_{W^{1,\infty}(\Omega_t)}) < +\infty,$$

$$\inf_{(t,x,y) \in [0,T] \times \Sigma_t} -\frac{\partial P}{\partial \mathbf{n}}(t, x, y) \geq c_0,$$

$$1 + \eta(t, x) \geq h_0 \quad \text{for } x \in \mathbf{R}^d,$$

for some $p > 2d$ and $c_0 > 0, h_0 > 0$, then it holds that

$$\sup_{t \in [0, T]} E_s(t) \leq C(E_s(0), M(T), T, c_0, h_0),$$

Especially, the solution (η, v) can be extended after $t = T$. Here $H(t, x)$ is the mean curvature of the free surface and

$$E_s(t) \stackrel{\text{def}}{=} \|\eta(t)\|_{H^{s+\frac{1}{2}}} + \|v(t)\|_{H^{s+\frac{1}{2}}(\Omega(t))}.$$

1.4. Main ideas. We denote by (V, B) the horizontal and vertical traces of the velocity on the free surface, i.e.,

$$V \triangleq (v^1, \dots, v^d)|_{y=\eta}, \quad B \triangleq v^{d+1}|_{y=\eta}.$$

Introduce a good unknown $U = V + T_\zeta B$, where $\zeta = \nabla \eta$ and T_ζ is Bony's paraproduct. We can derive the following evolution equation for U :

$$D_t^2 U + T_{a\lambda} U = f + f_\omega. \quad (1.9)$$

Here $D_t = \partial_t + T_V \cdot \nabla$, $T_{a\lambda}$ is an elliptic paradifferential operator of order one, and f_ω is the nonlinear term induced by the vorticity.

Compared with the irrotational case, a main difficulty is that f_ω lose one half derivative. More precisely, $f \in H^{s-\frac{1}{2}}$ but $f_\omega \in H^{s-1}$. Our key observation is that $D_t f_\omega$ has the same regularity as f_ω , and $\|f_\omega\|_{H^{s-1}}$ can be controlled by the lower order energy. By using the following trick

$$\begin{aligned} \langle \langle D \rangle^{s-\frac{1}{2}} f_\omega, \langle D \rangle^{s-\frac{1}{2}} D_t U \rangle &= \frac{d}{dt} \langle \langle D \rangle^{s-1} f_\omega, \langle D \rangle^s U \rangle - \langle \langle D \rangle^{s-1} D_t f_\omega, \langle D \rangle^s U \rangle \\ &\quad + \text{Lower order terms,} \end{aligned}$$

we can obtain an energy inequality of the form

$$E(t) \leq \langle \langle D \rangle^{s-1} f_\omega, \langle D \rangle^s U \rangle + E(0) + \int_0^t \mathcal{P}(E(t')) dt', \quad (1.10)$$

where \mathcal{P} is an increasing function. The term $\langle \langle D \rangle^{s-1} f_\omega, \langle D \rangle^s U \rangle$ can be controlled by

$$\|f_\omega\|_{H^{s-1}} \|U\|_{H^s} \leq \mathcal{P}(E_l(t)) E(t)^{\frac{1}{2}} \leq \mathcal{P}(E_l(t)) + \frac{1}{2} E(t).$$

Here $E_l(t)$ is a lower order energy, which satisfies

$$E_l(t) \leq E_l(0) + \int_0^t \mathcal{P}(E(t')) dt'.$$

This together with (1.10) gives a close estimate for $E(t)$.

Compared with the work [3], a new technical ingredient is that we introduce Chemin-Lerner type Besov spaces in the elliptic estimates such that we can obtain the maximal Hölder regularity estimates, which play an important role in the proof of break-down criterion. Otherwise, if we just follow the framework of [3], it is possible to establish a similar break-down criterion, where $\|v(t)\|_{W^{1,\infty}(\Omega_t)}$ is replaced by $\|v(t)\|_{C^{1,\alpha}(\Omega_t)}$ for some $\alpha > 0$.

For the free boundary problem, it is highly non trivial to obtain the existence of the solution from a priori estimates. The main reason is that many special structures of the system are used in the process of a priori estimates, however it is usually difficult to keep these structures for the approximate system.

To construct the approximate system, an immediate idea is that we use the equation (1.9) of U to construct the approximate system for the unknowns defined on the free surface. However, we find that it is difficult to show that the limit system is equivalent to the original Euler system. Instead, we still use the first order system to construct the iteration scheme. In order to keep as more structures of the system as possible, a key idea is that more unknowns and new equations are introduced such that the structures are integrated into the new equations. The construction of the iteration scheme are very tricky, where we also used another important observation: the maximal regularity of the free surface is not needed for the estimate of the vorticity and the velocity, and it is just used in the estimate of the pressure.

2. TOOLS OF PARADIFFERENTIAL OPERATORS

In this section, we introduce some basic results about the paradifferential operators from [24](see also [3]).

2.1. Paradifferential operators. Let us first introduce the definition of the symbol with limited spatial smoothness. We denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces for $k \in \mathbf{N}$, and the Hölder space with exponent k for $k \in (0, 1)$.

Definition 2.1. Given $\mu \in [0, 1]$ and $m \in \mathbf{R}$, we denote by $\Gamma_\mu^m(\mathbf{R}^d)$ the space of locally bounded functions $a(x, \xi)$ on $\mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \rightarrow \partial_\xi^\alpha a(x, \xi)$ belongs to $W^{\mu,\infty}$ and there exists a constant C_α such that

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\mu,\infty}} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|} \quad \text{for any } |\xi| \geq \frac{1}{2}.$$

The semi-norm of the symbol is defined by

$$M_\mu^m(a) \stackrel{\text{def}}{=} \sup_{|\alpha| \leq 3d/2+1+\mu} \sup_{|\xi| \geq 1/2} \|(1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\mu,\infty}}.$$

Especially, if a is a function independent of ξ , then

$$M_\mu^m(a) = \|a\|_{W^{\mu,\infty}}.$$

Given a symbol a , the paradifferential operator T_a is defined by

$$\widehat{T_a u}(\xi) \stackrel{\text{def}}{=} (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \psi(\eta) \widehat{u}(\eta) d\eta, \quad (2.1)$$

where $\widehat{a}(\theta, \xi)$ is the Fourier transform of a with respect to the first variable; the $\chi(\theta, \xi) \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ is an admissible cut-off function: there exists $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2$ and

$$\chi(\theta, \eta) = 1 \quad \text{for } |\theta| \leq \varepsilon_1 |\eta|, \quad \chi(\theta, \eta) = 0 \quad \text{for } |\theta| \geq \varepsilon_2 |\eta|,$$

and such that for any $(\theta, \eta) \in \mathbf{R}^d \times \mathbf{R}^d$,

$$|\partial_\theta^\alpha \partial_\eta^\beta \chi(\theta, \eta)| \leq C_{\alpha,\beta} (1 + |\eta|)^{-|\alpha|-|\beta|}.$$

The cut-off function $\psi(\eta) \in C^\infty(\mathbf{R}^d)$ satisfies

$$\psi(\eta) = 0 \quad \text{for } |\eta| \leq 1, \quad \psi(\eta) = 1 \quad \text{for } |\eta| \geq 2.$$

Here we will take the admissible cut-off function $\chi(\theta, \eta)$ as follows

$$\chi(\theta, \eta) = \sum_{k=0}^{\infty} \zeta_{k-3}(\theta) \varphi_k(\eta),$$

where $\zeta(\theta) = 1$ for $|\theta| \leq 1.1$ and $\zeta(\theta) = 0$ for $|\theta| \geq 1.9$; and

$$\begin{aligned}\zeta_k(\theta) &= \zeta(2^{-k}\theta) \quad \text{for } k \in \mathbf{Z}, \\ \varphi_0 &= \zeta, \quad \varphi_k = \zeta_k - \zeta_{k-1} \quad \text{for } k \geq 1.\end{aligned}$$

We also introduce the Littlewood-Paley operators Δ_k, S_k defined by

$$\begin{aligned}\Delta_k u &= \mathcal{F}^{-1}(\varphi_k(\xi)\widehat{u}(\xi)) \quad \text{for } k \geq 0, \quad \Delta_k u = 0 \quad \text{for } k < 0, \\ S_k u &= \sum_{\ell \leq k} \Delta_\ell u \quad \text{for } k \in \mathbf{Z}.\end{aligned}$$

In the case when the function a depends only on the first variable x in $T_a u$, we take $\psi = 1$. Then $T_a u$ is just the usual Bony's paraproduct defined by

$$T_a u = \sum_k S_{k-3} a \Delta_k u. \quad (2.2)$$

Furthermore, we have Bony's decomposition:

$$au = T_a u + T_u a + R(u, a), \quad (2.3)$$

where the remainder term $R(u, a)$ is defined by

$$R(u, a) = \sum_{|k-\ell| \leq 2} \Delta_k a \Delta_\ell u. \quad (2.4)$$

The following Bernstein's inequality will be repeatedly used.

Lemma 2.2. *Let $1 \leq p \leq q \leq \infty, \alpha \in \mathbf{N}^d$. Then it holds that*

$$\begin{aligned}\|\partial^\alpha S_k u\|_{L^q} &\leq C 2^{kd(\frac{1}{p}-\frac{1}{q}+|\alpha|)} \|S_k u\|_{L^p} \quad \text{for } k \in \mathbf{N}, \\ \|\Delta_k u\|_{L^q} &\leq C 2^{kd(\frac{1}{p}-\frac{1}{q}-|\alpha|)} \sup_{|\beta|=|\alpha|} \|\partial^\beta \Delta_k u\|_{L^p} \quad \text{for } k \geq 1.\end{aligned}$$

2.2. Functional spaces. We introduce some functional spaces, which will be used throughout this paper.

Definition 2.3. *Let $s \in \mathbf{R}$ and $p, q \in [1, \infty]$. The inhomogeneous Besov space $B_{p,q}^s(\mathbf{R}^d)$ is defined by*

$$B_{p,q}^s(\mathbf{R}^d) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{S}'(\mathbf{R}^d) : \|f\|_{B_{p,q}^s} \triangleq \left(\sum_j 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty \right\}.$$

In the case of $p = q = 2$, $B_{p,q}^s(\mathbf{R}^d)$ is just the usual Sobolev space $H^s(\mathbf{R}^d)$; In the case of $p = q = \infty$, $B_{p,q}^s(\mathbf{R}^d)$ is the Zygmund space $C^s(\mathbf{R}^d)$.

Let $\mathcal{S} = \mathbf{R}^d \times I$ with $I \subset \mathbf{R}$ an interval. We introduce the Sobolev space $H^s(\mathcal{S})$ on \mathcal{S} . When s is an integer, $H^s(\mathcal{S})$ is just the usual Sobolev space. In general case, let $k = [s]$ and $\sigma = s - k \in (0, 1)$. The norm of $H^s(\mathcal{S})$ is defined by

$$\|u\|_{H^s(\mathcal{S})} \stackrel{\text{def}}{=} \sum_{\ell \leq k} \|\nabla_{x,z}^\ell u\|_{L_z^2(I; H^\sigma)} + \left(\int_I \int_I \int_{\mathbf{R}^d} \frac{|\nabla_{x,z}^k u(x, z) - \nabla_{x,z}^k u(x, z')|^2}{|z - z'|^{1+2\sigma}} dx dz dz' \right)^{\frac{1}{2}}.$$

In order to obtain the optimal elliptic regularity, let us introduce Chemin-Lerner type Besov space $\widetilde{L}_z^q(I; B_{p,r}^s(\mathbf{R}^d))$, whose norm is defined by

$$\|f\|_{\widetilde{L}_z^q(I; B_{p,r}^s)} \stackrel{\text{def}}{=} \left(\sum_k 2^{ksr} \|\Delta_k f\|_{L_z^q(I; L^p)}^r \right)^{\frac{1}{r}}.$$

In the case of $p = r = \infty$, we denote it by $\tilde{L}_z^q(I; C^s(\mathbf{R}^d))$; In the case of $p = q = r = 2$, we have $\tilde{L}_z^q(I; B_{p,r}^s(\mathbf{R}^d)) \equiv L_z^2(I; H^s(\mathbf{R}^d))$; When $q = \infty, p = r = 2$, we denote it by $\tilde{L}_z^\infty(I; H^s(\mathbf{R}^d))$. In this case, there holds

$$\|f\|_{L_z^\infty(I; H^s)} \leq \|f\|_{\tilde{L}_z^\infty(I; H^s)}.$$

This kind of space was firstly introduced by Chemin and Lerner [13] to study the incompressible Navier-Stokes equations.

The following characterization of Sobolev space is very useful.

Lemma 2.4. *Let $s \in \mathbf{R}$ and $c > 0$. Suppose that $\{u_k\}_{k \in \mathbf{N}}$ is a sequence of functions in $L^2(\mathbf{R}^d)$ such that (1) u_0 is spectrally supported in a ball $\{|\xi| \leq c^{-1}\}$ and u_k for $k > 0$ is spectrally supported in an annulus $\{c2^k \leq |\xi| \leq c^{-1}2^k\}$; (2) $\{2^{ks}\|u_k\|_{L^2}\}_{k \in \mathbf{N}} \in \ell^2$. Then $u = \sum_k u_k \in H^s(\mathbf{R}^d)$ and*

$$\|u\|_{H^s} \leq C \left(\sum_k 2^{2ks} \|u_k\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

In addition, for $s > 0$, it is sufficient to assume that u_k is spectrally supported in a ball $\{|\xi| \leq c^{-1}2^k\}$.

We also introduce the anisotropic Sobolev space $H^{s,\sigma}(\mathbf{R}^{d+1})$, whose norm is given by

$$\|u\|_{H^{s,\sigma}} \stackrel{\text{def}}{=} \|\langle D_x \rangle^\sigma u\|_{H^s(\mathbf{R}^{d+1})}, \quad x \in \mathbf{R}^d.$$

We also have a similar characterization.

Lemma 2.5. *For any $s, \sigma \in \mathbf{R}$, we have*

$$\|u\|_{H^{s,\sigma}}^2 \sim \sum_{\ell,j} 2^{2\ell s} 2^{2j\sigma} \|\Delta_\ell \Delta_j^h u\|_{L^2}^2.$$

Here Δ_j^h is the Littlewood-Paley operator in the x direction. If $s, \sigma > 0$, and the sequence $\{u_{\ell,j}\}_{\ell,j \in \mathbf{N}}$ is spectrally supported in $\{|\xi| \leq c^{-1}2^\ell\} \cap \{|\xi_h| \leq c^{-1}2^j\}$ for some $c > 0$ and satisfies $\|u_{\ell,j}\|_{L^2} \leq c_{\ell,j} 2^{-\ell s} 2^{-j\sigma}$ with $\{c_{\ell,j}\}_{\ell,j \in \mathbf{N}} \in \ell^2$, then $u = \sum_{\ell,j} u_{\ell,j} \in H^{s,\sigma}(\mathbf{R}^{d+1})$ satisfies

$$\|u\|_{H^{s,\sigma}} \leq C \left(\sum_{\ell,j} 2^{2\ell s} 2^{2j\sigma} \|u_{\ell,j}\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

2.3. Symbolic calculus. Let us recall the symbolic calculus and the boundedness in Sobolev space and Besov space of the paradifferential operators.

Proposition 2.6. *Let $m, m' \in \mathbf{R}$.*

1. *If $a \in \Gamma_0^m(\mathbf{R}^d)$, then for any $s \in \mathbf{R}$,*

$$\|T_a\|_{H^s \rightarrow H^{s-m}} \leq CM_0^m(a).$$

2. *If $a \in \Gamma_\rho^m(\mathbf{R}^d), b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ for $\rho > 0$, then for any $s \in \mathbf{R}$,*

$$\|T_a T_b - T_{a \# b}\|_{H^s \rightarrow H^{s-m-m'+\rho}} \leq CM_\rho^{m_1}(a) M_0^{m'}(b) + CM_0^{m_1}(a) M_\rho^{m'}(b),$$

$$\text{where } a \# b = \sum_{|\alpha| < \rho} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi), D_x = \frac{\partial_x}{i}.$$

3. If $a \in \Gamma_\rho^m(\mathbf{R}^d)$ for $\rho \in (0, 1]$, then for any $s \in \mathbf{R}$,

$$\|T_{a^*} - (T_a)^*\|_{H^s \rightarrow H^{s-m+\rho}} \leq CM_\rho^m(a).$$

Here $(T_a)^*$ is the adjoint operator of T_a , and C is a constant independent of a, b .

Proposition 2.7. [32] Let $m, m', s \in \mathbf{R}, q \in [1, \infty]$ and $\rho \in [0, 1]$.

1. If $a \in \Gamma_0^m(\mathbf{R}^d)$, then

$$\|T_a\|_{B_{\infty,q}^s \rightarrow B_{\infty,q}^{s-m}} \leq CM_0^m(a);$$

2. If $a \in \Gamma_\rho^m(\mathbf{R}^d), b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$, then

$$\|T_a T_b - T_{ab}\|_{B_{\infty,q}^s \rightarrow B_{\infty,q}^{s-m-m'+\rho}} \leq CM_\rho^m(a)M_0^{m'}(b) + CM_0^m(a)M_\rho^{m'}(b).$$

Here C is a constant independent of a, b .

Remark 2.8. If the symbol $a(x, \xi)$ satisfies

$$M_{-\mu}^m(a) \triangleq \sup_{|\alpha| \leq 3d/2+1} \sup_{|\xi| \geq 1/2} \|(1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot, \xi)\|_{C^{-\mu}} < \infty$$

for some $\mu > 0$, then T_a is bounded from $H^s(\mathbf{R}^d)$ to $H^{s-m-\mu}(\mathbf{R}^d)$ and $B_{\infty,q}^s(\mathbf{R}^d)$ to $B_{\infty,q}^{s-m-\mu}(\mathbf{R}^d)$ with the bound $M_{-\mu}^m(a)$.

Corollary 2.9. Let $s, m_1, m_2, m_3 \in \mathbf{R}$. Suppose that $a \in \Gamma_1^{m_1}, b \in \Gamma_2^{m_2}, c \in \Gamma_2^{m_3}$. Then we have

$$\|[T_a, [T_b, T_c]]\|_{H^s \rightarrow H^{s-m_1-m_2-m_3+2}} \leq CM_1^{m_1}(a)M_2^{m_2}(b)M_2^{m_3}(c).$$

Proof. It follows from Proposition 2.6 that

$$\|[T_b, T_c] - T_p + T_{p_1}\|_{H^s \rightarrow H^{s-m_2-m_3+2}} \leq M_2^{m_2}(b)M_2^{m_3}(c).$$

where $p(x, \xi) = \sum_{|\alpha|=1} \partial_\xi^\alpha b(x, \xi) D_x^\alpha c(x, \xi)$ and $p_1(x, \xi) = \sum_{|\alpha|=1} \partial_\xi^\alpha c(x, \xi) D_x^\alpha b(x, \xi)$. Hence, it is sufficient to consider $[T_a, T_p]$ and $[T_a, T_{p_1}]$. Because of $p, p_1 \in \Gamma_1^{m_2+m_3-1}$, the corollary follows from Proposition 2.6. \square

2.4. Tame estimates in Sobolev space. Let us first recall some classical tame estimates. One can refer to [8] for more general results.

Lemma 2.10. Let $s \in \mathbf{R}$, and $p, q \in [1, \infty]$. Then for any $\sigma > 0$, we have

$$\|T_u v\|_{B_{p,q}^s} \leq C \min(\|u\|_{C^{-\sigma}} \|v\|_{B_{p,q}^{s+\sigma}}, \|u\|_{L^\infty} \|v\|_{B_{p,q}^s}).$$

If $s_1 + s_2 > 0$, then we have

$$\|R(u, v)\|_{H^{s_1+s_2-\frac{d}{2}}} \leq C \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.$$

If $s > 0$, then for any $\sigma \in \mathbf{R}$, we have

$$\|R(u, v)\|_{B_{p,q}^s} \leq C \|u\|_{C^\sigma} \|v\|_{B_{p,q}^{s-\sigma}}.$$

Proof. The first two inequalities are classical, see [8] for example. We prove the third inequality. Recall that

$$R(u, v) = \sum_{|k-\ell| \leq 2} \Delta_k u \Delta_\ell v.$$

Hence, there exists some $N_0 \in \mathbf{N}$ so that

$$\Delta_j R(u, v) = \sum_{|k-\ell| \leq 2; k, \ell \geq j-N_0} \Delta_j (\Delta_k u \Delta_\ell v).$$

It follows from Lemma 2.2 that

$$\begin{aligned} \|\Delta_j R(u, v)\|_{L^p} &\leq \sum_{|k-\ell| \leq 2; k, \ell \geq j-N_0} \|\Delta_k u\|_{L^\infty} \|\Delta_\ell v\|_{L^p} \\ &\leq C \|u\|_{C^\sigma} \sum_{\ell \geq j-N_0} 2^{-\sigma \ell} \|\Delta_\ell v\|_{L^p}. \end{aligned}$$

This gives

$$2^{js} \|\Delta_j R(u, v)\|_{L^p} \leq C \|u\|_{C^\sigma} \sum_{\ell \geq j-N_0} 2^{-s(\ell-j)} 2^{\ell(s-\sigma)} \|\Delta_\ell v\|_{L^p},$$

which implies the second inequality by Young's inequality. \square

A direct consequence of Lemma 2.10 is the following tame product estimate.

Lemma 2.11. *Let $s \geq 0$. Then we have*

$$\begin{aligned} \|fg\|_{H^s} &\leq C(\|f\|_{L^\infty} \|g\|_{H^s} + \|g\|_{L^\infty} \|f\|_{H^s}), \\ \|f \nabla g\|_{H^s} &\leq C(\|f\|_{L^\infty} \|\nabla g\|_{H^s} + \|g\|_{L^\infty} \|f\|_{H^{s+1}}). \end{aligned}$$

Lemma 2.12. [8] *Let $s > 0, p, q \in [1, \infty]$ and F be a smooth function with $F(0) = 0$. Then we have*

$$\|F(u)\|_{B_{p,q}^s} \leq C(\|u\|_{L^\infty}) \|u\|_{B_{p,q}^s}.$$

Especially, for $p = q = 2$, we have

$$\|F(u)\|_{H^s} \leq C(\|u\|_{L^\infty}) \|u\|_{H^s}.$$

Using an extension argument, we deduce from Lemma 2.11 and Lemma 2.12 that

Lemma 2.13. *Let $\mathcal{S} = \mathbf{R}^d \times I$ with $I \subset \mathbf{R}$ an interval and $s \geq 0$. Then we have*

$$\begin{aligned} \|uv\|_{H^s(\mathcal{S})} &\leq C(\|u\|_{L^\infty(\mathcal{S})} \|v\|_{H^s(\mathcal{S})} + \|v\|_{L^\infty(\mathcal{S})} \|u\|_{H^s(\mathcal{S})}), \\ \|u \nabla v\|_{H^s(\mathcal{S})} &\leq C(\|u\|_{L^\infty(\mathcal{S})} \|\nabla v\|_{H^s(\mathcal{S})} + \|v\|_{L^\infty(\mathcal{S})} \|u\|_{H^{s+1}(\mathcal{S})}). \end{aligned}$$

Let F be a smooth function with $F(0) = 0$. Then we have

$$\|F(u)\|_{H^s(\mathcal{S})} \leq C(\|u\|_{L^\infty(\mathcal{S})}) \|u\|_{H^s(\mathcal{S})}.$$

Next we present a tame estimate in the anisotropic Sobolev space.

Lemma 2.14. *Let $s > 0$ and $\sigma \in (0, 1)$. Then it holds that for any $\epsilon > 0$,*

$$\begin{aligned} &\|T_u v\|_{H^{s,\sigma}} + \|R(u, v)\|_{H^{s,\sigma}} \\ &\leq C(\|u\|_{L^\infty} + \|\langle D_x \rangle^{\sigma+\epsilon} u\|_{L_y^\infty(\mathbf{R}; L^2)} + \|\langle D_x \rangle^{\frac{d}{2}} u\|_{L_y^\infty(\mathbf{R}; L^2)}) \|v\|_{H^{s,\sigma}}. \end{aligned}$$

Proof. Using Bony's decomposition (2.3), we write

$$T_u v = T(T^h + \bar{T}^h + R^h)(u, v).$$

where $T(u, v) = T_u v$, $\bar{T}^h(u, v) = T^h(v, u)$, and $T_u^h v$ denote the paraproduct in the x direction.

By using Lemma 2.2 and Lemma 2.5, it is easy to show that

$$\|(TT^h + TR^h)(u, v)\|_{H^{s,\sigma}} \leq C\|u\|_{L^\infty} \|v\|_{H^{s,\sigma}}. \quad (2.5)$$

According to the definition of the paraproduct, we have

$$(T\overline{T}^h)(u, v) = \sum_{\ell, j} S_{\ell-3} \Delta_j^h u \Delta_\ell S_{j-3}^h v.$$

For $\sigma < \frac{d}{2}$, we get by Lemma 2.2 that

$$\begin{aligned} \|S_{\ell-3} \Delta_j^h u \Delta_\ell S_{j-3}^h v\|_{L^2(\mathbf{R}^{d+1})} &\leq \|S_{\ell-3} \Delta_j^h u\|_{L_y^\infty(\mathbf{R}; L^2)} \|\Delta_\ell S_{j-3}^h v\|_{L_y^2(\mathbf{R}; L^\infty)} \\ &\leq C 2^{j(\frac{d}{2}-\sigma)} \|\Delta_j^h u\|_{L_y^\infty(\mathbf{R}; L^2)} \|\langle D_x \rangle^\sigma \Delta_\ell v\|_{L^2} \\ &\leq C 2^{-j\sigma} \|\langle D_x \rangle^{\frac{d}{2}} \Delta_j^h u\|_{L_y^\infty(\mathbf{R}; L^2)} \|\langle D_x \rangle^\sigma \Delta_\ell v\|_{L^2} \\ &\leq C c_{\ell, j} 2^{-\ell s} 2^{-j\sigma} \|\langle D_x \rangle^{\frac{d}{2}} u\|_{L_y^\infty(\mathbf{R}; L^2)} \|v\|_{H^{s, \sigma}}, \end{aligned}$$

and for $\sigma \geq \frac{d}{2}$ and any $\epsilon > 0$, we have

$$\begin{aligned} \|S_{\ell-3} \Delta_j^h u \Delta_\ell S_{j-3}^h v\|_{L^2(\mathbf{R}^{d+1})} &\leq \|S_{\ell-3} \Delta_j^h u\|_{L_y^\infty(\mathbf{R}; L^2)} \|\Delta_\ell S_{j-3}^h v\|_{L_z^2(\mathbf{R}; L^\infty)} \\ &\leq C 2^{j\epsilon} \|\Delta_j^h u\|_{L_y^\infty(\mathbf{R}; L^2)} \|\langle D_x \rangle^{\frac{d}{2}-\epsilon} \Delta_\ell v\|_{L^2} \\ &\leq C c_{\ell, j} 2^{-\ell s} 2^{-j\sigma} \|\langle D_x \rangle^{\sigma+\epsilon} u\|_{L_y^\infty(\mathbf{R}; L^2)} \|v\|_{H^{s, \sigma}}, \end{aligned}$$

where $\|\{c_{\ell, j}\}\|_{\ell^2} \leq 1$. Then Lemma 2.5 ensures that

$$\|(T\overline{T}^h)(u, v)\|_{H^{s, \sigma}} \leq C(\|\langle D_x \rangle^{\sigma+\epsilon} u\|_{L_y^\infty(\mathbf{R}; L^2)} + \|\langle D_x \rangle^{\frac{d}{2}} u\|_{L_y^\infty(\mathbf{R}; L^2)}) \|v\|_{H^{s, \sigma}},$$

which together with (2.5) gives the first part of the lemma. The proof of another part is similar. \square

Next, we give the material derivative estimates for $R(u, v)$.

Lemma 2.15. *Let $\overline{\partial}_t \triangleq \partial_t + V \cdot \nabla$. Let $s > 0$ and $\sigma_1 \in (0, 1), \sigma_2 \in \mathbf{R}$ and $\sigma > 1 - \sigma_1$. Then it holds that*

$$\begin{aligned} \|\overline{\partial}_t R(u(t), v(t))\|_{H^s} &\leq C(\|\overline{\partial}_t u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}} + \|\overline{\partial}_t v\|_{C^{\sigma_2}} \|u\|_{H^{s-\sigma_2}} \\ &\quad + \|\nabla V\|_{L^\infty} (\|u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}} + \|v\|_{L^\infty} \|u\|_{H^s}) + \|u\|_{C^{\sigma_1}} \|v\|_{L^\infty} \|V\|_{H^{s+\sigma}}), \end{aligned}$$

where $\|\overline{\partial}_t u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}}$ and $\|u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}}$ can also be replaced by $\|\overline{\partial}_t u\|_{L^\infty} \|v\|_{H^s}$ and $\|u\|_{L^\infty} \|v\|_{H^s}$ respectively.

Proof. A direct calculation gives

$$\begin{aligned} \overline{\partial}_t R(u, v) &= \sum_{|k-\ell| \leq 2} \overline{\partial}_t (\Delta_k u \Delta_\ell V) \\ &= \sum_{|k-\ell| \leq 2} \left(\Delta_k (\overline{\partial}_t u) \Delta_\ell v + \Delta_k u \Delta_\ell (\overline{\partial}_t v) \right. \\ &\quad \left. - [\Delta_k, V] \cdot \nabla u \Delta_\ell v - \Delta_k u [\Delta_\ell, V] \cdot \nabla v \right) \\ &= R(\overline{\partial}_t u, v) + R(u, \overline{\partial}_t v) - \sum_{|k-\ell| \leq 2} ([\Delta_k, V] \cdot \nabla u \Delta_\ell v + \Delta_k u [\Delta_\ell, V] \cdot \nabla v). \end{aligned}$$

It follows from Lemma 2.10 that

$$\begin{aligned} \|R(\overline{\partial}_t u, v)\|_{H^s} &\leq C \|\overline{\partial}_t u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}}, \\ \|R(u, \overline{\partial}_t v)\|_{H^s} &\leq C \|\overline{\partial}_t v\|_{C^{\sigma_2}} \|u\|_{H^{s-\sigma_2}}. \end{aligned}$$

Given $N_0 \in \mathbf{N}$ sufficiently large, we further decompose

$$\begin{aligned} \sum_{|k-\ell|\leq 2} [\Delta_k, V] \cdot \nabla u \Delta_\ell v &= \sum_{|k-\ell|\leq 2} [\Delta_k, S_{k-N_0} V] \cdot \nabla u \Delta_\ell v \\ &\quad + \sum_{|k-\ell|\leq 2} [\Delta_k, S^{k-N_0} V] \cdot \nabla u \Delta_\ell v, \end{aligned}$$

where $S^k = 1 - S_k$. By noting that u has to be spectrally supported in $\{|\xi| \sim 2^k\}$ and using the commutator estimate

$$\|[\Delta_k, u] \nabla v\|_{L^p} \leq C \|\nabla u\|_{L^\infty} \|v\|_{L^p} \quad \text{for any } p \in [1, \infty], \quad (2.6)$$

which follows from the identity

$$\begin{aligned} [\Delta_k, u] \nabla v(x) &= \int_{\mathbf{R}^d} \check{\varphi}_k(x-x')(u(x') - u(x)) \nabla v(x') dx' \\ &= \int_{\mathbf{R}^d} \nabla \check{\varphi}_k(x-x')(u(x') - u(x)) v(x') dx' \\ &\quad - \int_{\mathbf{R}^d} \check{\varphi}_k(x-x') \nabla u(x') v(x') dx', \end{aligned}$$

and $\|\check{\varphi}_k\|_{L^1} + \|x \nabla \check{\varphi}_k\|_{L^1} \leq C$, then we can deduce that

$$\begin{aligned} \|[\Delta_k, S_{k-N_0} V] \cdot \nabla u \Delta_\ell v\|_{L^2} &\leq C 2^{-k\sigma_1} \|\nabla V\|_{L^\infty} \|u\|_{C^{\sigma_1}} \|\Delta_\ell v\|_{L^2} \\ &\leq C c_\ell 2^{-\ell s} \|\nabla V\|_{L^\infty} \|u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}} \end{aligned}$$

with $\|\{c_\ell\}\|_{\ell^2} \leq 1$. Obviously, the term $[\Delta_k, S_{k-N_0} V] \cdot \nabla u \Delta_\ell v$ is spectrally supported in a ball $\{|\xi| \lesssim 2^k\}$ for $|k-\ell| \leq 2$. Then Lemma 2.4 ensures that

$$\left\| \sum_{|k-\ell|\leq 2} [\Delta_k, S_{k-N_0} V] \cdot \nabla u \Delta_\ell v \right\|_{H^s} \leq C \|\nabla V\|_{L^\infty} \|u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}}.$$

Noticing that

$$\begin{aligned} \|\Delta_k(S^{k-N_0} V \cdot \nabla u) \Delta_\ell v\|_{L^2} &\leq \|\Delta_k(S^{k-N_0} V \cdot \nabla S_k u) \Delta_\ell v\|_{L^2} \\ &\quad + \|\Delta_k(S^{k-N_0} \nabla \cdot V S^k u) \Delta_\ell v\|_{L^2} + \|\Delta_k \nabla \cdot (S^{k-N_0} V S^k u) \Delta_\ell v\|_{L^2} \\ &\leq C 2^{-\sigma_1 k} \|\nabla V\|_{L^\infty} \|u\|_{C^{\sigma_1}} \|\Delta_\ell v\|_{L^2}. \end{aligned}$$

we can deduce from Lemma 2.4 that

$$\left\| \sum_{|k-\ell|\leq 2} \Delta_k(S^{k-N_0} V \cdot \nabla u) \Delta_\ell v \right\|_{H^s} \leq C \|\nabla V\|_{L^\infty} \|u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}}.$$

On the other hand, we have

$$\begin{aligned} \left\| \sum_{|k-\ell|\leq 2} S^{k-N_0} V \cdot \nabla \Delta_k u \Delta_\ell v \right\|_{H^s} &\leq C \sum_{|k-\ell|\leq 2} \|S^{k-N_0} V \cdot \nabla \Delta_k u \Delta_\ell v\|_{H^s} \\ &\leq C \sum_{|k-\ell|\leq 2} 2^{(1-\sigma_1)k} \|S^{k-N_0} V\|_{H^s} \|u\|_{C^{\sigma_1}} \|v\|_{L^\infty} \\ &\leq C \sum_{|k-\ell|\leq 2} 2^{-k(\sigma-1+\sigma_1)} \|V\|_{H^{s+\sigma}} \|u\|_{C^{\sigma_1}} \|v\|_{L^\infty} \\ &\leq C \|V\|_{H^{s+\sigma}} \|u\|_{C^{\sigma_1}} \|v\|_{L^\infty}. \end{aligned}$$

This shows that

$$\begin{aligned} \left\| \sum_{|k-\ell|\leq 2} [\Delta_k, S^{k-N_0} V] \cdot \nabla u \Delta_\ell v \right\|_{H^s} &\leq C (\|\nabla V\|_{L^\infty} \|u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}} \\ &\quad + \|V\|_{H^{s+\sigma}} \|u\|_{C^{\sigma_1}} \|v\|_{L^\infty}). \end{aligned}$$

Obviously, $\|u\|_{C^{\sigma_1}} \|v\|_{H^{s-\sigma_1}}$ can be replaced by $\|u\|_{L^\infty} \|v\|_{H^s}$ in the above proof.

In a similar way, we can deduce that

$$\begin{aligned} \left\| \sum_{|k-\ell|\leq 2} \Delta_k u [\Delta_\ell, V] \cdot \nabla v \right\|_{H^s} &\leq C (\|\nabla V\|_{L^\infty} \|v\|_{L^\infty} \|u\|_{H^s} \\ &\quad + \|V\|_{H^{s+\sigma}} \|v\|_{L^\infty} \|u\|_{C^{\sigma_1}}). \end{aligned}$$

This completes the proof of the lemma. \square

2.5. Tame estimates in Chemin-Lerner spaces. Let us recall the following lemmas from [32].

Lemma 2.16. *Let $s \in \mathbf{R}$ and $q, q_1, q_2 \in [1, \infty]$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then for any $s_1 > 0$, we have*

$$\|T_g f\|_{\tilde{L}_z^q(I; H^s)} \leq C \min (\|g\|_{\tilde{L}_z^{q_1}(I; L^\infty)} \|f\|_{\tilde{L}_z^{q_2}(I; H^s)}, \|g\|_{\tilde{L}_z^{q_1}(I; C^{-s_1})} \|f\|_{\tilde{L}_z^{q_2}(I; H^{s+s_1})}).$$

Lemma 2.17. *Let $s \in \mathbf{R}$ and $q, q_1, q_2, r \in [1, \infty]$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then for any $s_1 > 0$, we have*

$$\|T_g f\|_{\tilde{L}_z^q(I; B_{\infty, r}^s)} \leq C \min (\|g\|_{\tilde{L}_z^{q_1}(I; L^\infty)} \|f\|_{\tilde{L}_z^{q_2}(I; B_{\infty, r}^s)}, \|g\|_{\tilde{L}_z^{q_1}(I; C^{-s_1})} \|f\|_{\tilde{L}_z^{q_2}(I; B_{\infty, r}^{s+s_1})}).$$

Lemma 2.18. *Let $q, q_1, q_2, r \in [1, \infty]$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then for any $s > 0$ and $s_1 \in R$, we have*

$$\begin{aligned} \|R(f, g)\|_{\tilde{L}_z^q(I; H^s)} &\leq C \|g\|_{\tilde{L}_z^{q_1}(I; C^{s_1})} \|f\|_{\tilde{L}_z^{q_2}(I; H^{s-s_1})}, \\ \|R(f, g)\|_{\tilde{L}_z^q(I; B_{\infty, r}^s)} &\leq C \|g\|_{\tilde{L}_z^{q_1}(I; C^{s_1})} \|f\|_{\tilde{L}_z^{q_2}(I; B_{\infty, r}^{s-s_1})}. \end{aligned}$$

If $s \leq 0$ and $s_1 + s_2 > 0$, then we have

$$\begin{aligned} \|R(f, g)\|_{\tilde{L}_z^q(I; H^s)} &\leq C \|g\|_{\tilde{L}_z^{q_1}(I; C^{s_1})} \|f\|_{\tilde{L}_z^{q_2}(I; H^{s_2})}, \\ \|R(f, g)\|_{\tilde{L}_z^q(I; B_{\infty, r}^s)} &\leq C \|g\|_{\tilde{L}_z^{q_1}(I; C^{s_1})} \|f\|_{\tilde{L}_z^{q_2}(I; C^{s_2})}. \end{aligned}$$

2.6. Commutator estimates.

Lemma 2.19. *Let $m, \mu \in \mathbf{R}$, $s > 0$ and $a \in \Gamma_\rho^m(\mathbf{R}^d)$ with $\rho \in (0, 1]$. Then there holds*

$$\|[\langle D \rangle^s, T_a]u\|_{H^\mu} \leq C M_\rho^m(a) \|u\|_{H^{s+\mu+m-\rho}}.$$

Proof. We write

$$[\langle D \rangle^s, T_a]u = T_{\langle \xi \rangle^s} T_a u - T_{\langle \xi \rangle^s a} u + (\langle D \rangle^s - T_{\langle \xi \rangle^s}) T_a u,$$

then the proposition follows from Proposition 2.6 and the fact that

$$\|(\langle D \rangle^s - T_{\langle \xi \rangle^s}) T_a u\|_{H^\mu} \leq C \|\langle D \rangle^s (1 - \psi(D)) T_a u\|_{H^\mu} \leq C M_0^m(a) \|u\|_{H^{-\mu'}}$$

for any $\mu' > 0$. \square

Lemma 2.20. *Let $\mathcal{S} = \mathbf{R}^d \times I$ with $I \subset \mathbf{R}$ an interval. Then it holds that for any integer $s \geq 1$,*

$$\|[\nabla_{x,y}^s, v] \nabla_{x,y} u\|_{L^2(\mathcal{S})} \leq C(\|\nabla_{x,y} v\|_{L^\infty(\mathcal{S})} \|u\|_{H^s(\mathcal{S})} + \|u\|_{L^\infty(\mathcal{S})} \|\nabla_{x,y} v\|_{H^s(\mathcal{S})}).$$

Proof. When $\mathcal{S} = \mathbf{R}^{d+1}$, this lemma is classical. General case can be deduced by using an extension argument. \square

Proposition 2.21. *Let $V \in C([0, T]; B_{\infty,1}^1(\mathbf{R}^d))$ and $p = p(t, x, \xi)$ be homogenous in ξ of order m . Then for any $s \geq 0$,*

$$\begin{aligned} & \| [T_p, \partial_t + T_V \cdot \nabla] u(t) \|_{H^s} \\ & \leq C \left(M_0^m(p) \|V(t)\|_{B_{\infty,1}^1} + M_0^m((\partial_t + T_V \cdot \nabla)p) \right) \|u(t)\|_{H^{s+m}}. \end{aligned}$$

If $p(t) \in \Gamma_\mu^m$ for some $\mu > 0$, $M_0^m(p) \|V(t)\|_{B_{\infty,1}^1}$ can be replaced by $M_\mu^m(p) \|V(t)\|_{W^{1,\infty}}$ in the case of $s > 0$.

Proof. The proof is motivated by Lemma 2.16 in [3]. As in [3], it suffices to consider the case when $p = p(t, x)$ by decomposing p into a sum of spherical harmonic. In this case, $M_0^0(p) = \|p\|_{L^\infty}$. A direct calculation gives

$$\begin{aligned} [\partial_t + T_V \cdot \nabla, T_p] u &= T_{\partial_t p} u + T_V \cdot T_{\nabla p} u + T_V \cdot T_p \nabla u - T_p T_V \cdot \nabla u \\ &= T_{(\partial_t + T_V \cdot \nabla)p} u - (T_{T_V \cdot \nabla p} - T_V \cdot T_{\nabla p}) u \\ &\quad + (T_V \cdot T_p \nabla u - T_p T_V \cdot \nabla u). \end{aligned}$$

We infer from Lemma 2.10 that

$$\|T_{(\partial_t + T_V \cdot \nabla)p} u\|_{H^s} \leq C \|(\partial_t + T_V \cdot \nabla)p\|_{L^\infty} \|u\|_{H^s}. \quad (2.7)$$

Recalling the definition $S^j u = u - S_j u$, we decompose $T_V \cdot T_{\nabla p} u$ as

$$\begin{aligned} T_V \cdot T_{\nabla p} u &= \sum_{j,k} S_{j-3} V \cdot \Delta_j (S_{k-3}(\nabla p) \Delta_k u) \\ &= \sum_k S_{k-3} V \cdot S_{k-3}(\nabla p) \Delta_k u + \sum_{j,k} (S_{j-3} V - S_{k-3} V) \cdot \Delta_j (S_{k-3}(\nabla p) \Delta_k u) \\ &= \sum_k S_{k-3} (V \cdot \nabla p) \Delta_k u - \sum_k S_{k-3} (S^{k-3} V \cdot \nabla p) \Delta_k u \\ &\quad - \sum_k [S_{k-3}, S_{k-3} V] \cdot \nabla p \Delta_k u + \sum_{j,k} (S_{j-3} V - S_{k-3} V) \cdot \Delta_j (S_{k-3}(\nabla p) \Delta_k u) \\ &\triangleq T_{V \cdot \nabla p} u + I_1 + I_2 + I_3. \end{aligned}$$

We get by Lemma 2.2 that

$$\begin{aligned} & \|(S_{j-3} V - S_{k-3} V) \cdot \Delta_j (S_{k-3}(\nabla p) \Delta_k u)\|_{L^2} \\ & \leq C 2^{-k(s-1)} \|S_{j-3} V - S_{k-3} V\|_{L^\infty} \|p\|_{L^\infty} \|u\|_{H^s}. \end{aligned}$$

Note that the summation index (j, k) in I_3 satisfies $|k - j| \leq N_0$ for some $N_0 \in \mathbf{N}$, and $(S_{j-3} V - S_{k-3} V) \cdot \Delta_j (S_{k-3}(\nabla p) \Delta_k u)$ is spectrally supported in a ball $\{|\xi| \lesssim 2^j\}$. Then Lemma 2.4 ensures that for $s > 0$,

$$\|I_3\|_{H^s} \leq C \|V\|_{W^{1,\infty}} \|p\|_{L^\infty} \|u\|_{H^s}$$

by using the fact that

$$\sum_{|j-k| \leq N_0} 2^k \|S_{j-3}V - S_{k-3}V\|_{L^\infty} \leq C \|V\|_{W^{1,\infty}}.$$

For $s = 0$, we have

$$\|I_3\|_{L^2} \leq C \|V\|_{B_{\infty,1}^1} \|p\|_{L^\infty} \|u\|_{L^2}.$$

By Lemma 2.2 again, we have

$$\begin{aligned} \|S_{k-3}(S^{k-3}V \cdot \nabla p) \Delta_k u\|_{L^2} &\leq C 2^{-ks} c_k (\|S^{k-3}(\nabla V)\|_{L^\infty} + 2^k \|S^{k-3}V\|_{L^\infty}) \|p\|_{L^\infty} \|u\|_{H^s} \\ &\leq C 2^{-ks} c_k \|V\|_{W^{1,\infty}} \|p\|_{L^\infty} \|u\|_{H^s}, \end{aligned}$$

with $\|\{c_k\}\|_{\ell^2} \leq 1$, and $S_{k-3}(S^{k-3}V \cdot \nabla p) \Delta_k u$ is spectrally supported in an annulus $\{|\xi| \sim 2^k\}$. Then Lemma 2.4 ensures that

$$\|I_1\|_{H^s} \leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s}.$$

Noticing that $[S_{k-3}, S_{k-3}V] \cdot \nabla p \Delta_k u$ is spectrally supported in a ball $\{|\xi| \lesssim 2^k\}$ and by (2.6),

$$\|[S_{k-3}, S_{k-3}V] \cdot \nabla p \Delta_k u\|_{L^2} \leq C c_k 2^{-ks} \|V\|_{W^{1,\infty}} \|p\|_{L^\infty} \|u\|_{H^s},$$

with $\|\{c_k\}\|_{\ell^2} \leq 1$, we infer from Lemma 2.4 that for $s > 0$,

$$\|I_2\|_{H^s} \leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s}.$$

In the case of $s = 0$, we need to decompose I_2 as

$$I_2 = \sum_k [S_{k-3}, S_{k-N_0}V] \cdot \nabla p \Delta_k u + \sum_k [S_{k-3}, (S_{k-3} - S_{k-N_0})V] \cdot \nabla p \Delta_k u,$$

where we take N_0 big enough so that $[S_{k-3}, S_{k-N_0}V] \cdot \nabla p \Delta_k u$ is spectrally supported in an annulus $\{|\xi| \sim 2^j\}$. Then we have

$$\|I_2\|_{L^2} \leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{L^2}.$$

Putting the estimates of I_1, I_2 and I_3 together, we deduce that

$$\begin{aligned} \|(T_V \cdot T_{\nabla p} - T_{V \cdot \nabla p})u\|_{H^s} &\leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s} (s > 0), \\ \|(T_V \cdot T_{\nabla p} - T_{V \cdot \nabla p})u\|_{L^2} &\leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{L^2}. \end{aligned}$$

By Bony's decomposition (2.3), we have

$$\begin{aligned} \|T_{V \cdot \nabla p} - T_{V \cdot \nabla p} u\|_{H^s} &\leq \|V \cdot \nabla p - T_V \cdot \nabla p\|_{L^\infty} \|u\|_{H^s} \\ &\leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{H^s}, \end{aligned}$$

and if $p \in C^\mu$, we have

$$\|T_{V \cdot \nabla p} - T_{V \cdot \nabla p} u\|_{H^s} \leq C \|p\|_{C^\mu} \|V\|_{W^{1,\infty}} \|u\|_{H^s}.$$

This shows that for $s > 0$

$$\|(T_{T_V \cdot \nabla p} - T_V \cdot T_{\nabla p})u\|_{H^s} \leq C \min(\|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1}, \|p\|_{C^\mu} \|V\|_{W^{1,\infty}}) \|u\|_{H^s}, \quad (2.8)$$

and for $s = 0$,

$$\|(T_{T_V \cdot \nabla p} - T_V \cdot T_{\nabla p})u\|_{H^s} \leq \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{H^s}. \quad (2.9)$$

Next we decompose $T_V \cdot T_p \nabla u$ as

$$\begin{aligned}
T_V \cdot T_p \nabla u &= \sum_{j,k} S_{j-3} V \Delta_j (S_{k-3} p \cdot \Delta_k \nabla u) \\
&= \sum_k S_{k-3} V S_{k-3} p \cdot \Delta_k \nabla u \\
&\quad + \sum_{j,k} (S_{j-3} V - S_{k-3} V) \Delta_j (S_{k-3} p \cdot \Delta_k \nabla u) \\
&\triangleq \sum_k S_{k-3} V S_{k-3} p \cdot \Delta_k \nabla u + II_1.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
T_p T_V \cdot \nabla u &= \sum_j S_{j-3} p \Delta_j (S_{j-3} V \cdot \nabla u) \\
&\quad + \sum_{j,k} S_{j-3} p \Delta_j ((S_{k-3} - S_{j-3}) V \cdot \nabla \Delta_k u) \\
&= \sum_j S_{j-3} V S_{j-3} p \cdot \Delta_j \nabla u + \sum_j S_{j-3} p [\Delta_j, S_{j-3} V] \cdot \nabla u \\
&\quad + \sum_{j,k} S_{j-3} p \cdot \Delta_j ((S_{k-3} - S_{j-3}) V \cdot \nabla \Delta_k u) \\
&\triangleq \sum_j S_{j-3} V S_{j-3} p \cdot \nabla \Delta_j u + II_2 + II_3.
\end{aligned}$$

This gives

$$T_V \cdot T_p \nabla u - T_p T_V \cdot \nabla u = II_1 - II_2 - II_3.$$

Notice that the summation index (j, k) in II_3 satisfies $|k-j| \leq N_0$ for some $N_0 \in \mathbb{N}$. Similar to I_3 , we can deduce that

$$\|II_3\|_{H^s} \leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s}.$$

For II_1 , we have

$$\begin{aligned}
\|II_1\|_{H^s} &\leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s} (s > 0), \\
\|II_1\|_{L^2} &\leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{L^2}.
\end{aligned}$$

We decompose II_2 as

$$\begin{aligned}
II_2 &= \sum_j S_{j-3} p (\Delta_j (T_{S_{j-3} V} \cdot \nabla u) - T_{S_{j-3} V} \cdot \nabla \Delta_j u) \\
&\quad + \sum_j S_{j-3} p \Delta_j (S_{j-3} V \cdot \nabla u - T_{S_{j-3} V} \cdot \nabla u) \\
&\quad - \sum_j S_{j-3} p (S_{j-3} V \cdot \nabla \Delta_j u - T_{S_{j-3} V} \cdot \nabla \Delta_j u) \\
&\triangleq II_2^1 + II_2^2 + II_2^3.
\end{aligned}$$

Similar to I_2 , we have

$$\begin{aligned}
\|II_2^1\|_{H^s} &\leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s} (s > 0), \\
\|II_2^1\|_{L^2} &\leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{L^2}.
\end{aligned}$$

Using Bony's decomposition (2.3), we decompose II_2^2 as

$$II_2^2 = \sum_j S_{j-3} p \Delta_j (R(S_{j-3} V, \nabla u) + T_{S^{j-N_0} \nabla u} \cdot S_{j-3} V)$$

for some $N_0 \in \mathbf{N}$. It is easy to prove that for $s > 0$,

$$\|S_{j-3} p \Delta_j (R(S_{j-3} V, \nabla u) + T_{S^{j-N_0} \nabla u} \cdot S_{j-3} V)\|_{L^2} \leq C c_j 2^{-js} \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s}$$

with $\|\{c_j\}\|_{\ell^2} \leq 1$. Then Lemma 2.4 ensures that for $s > 0$,

$$\|II_2^2\|_{H^s} \leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s}.$$

For $s = 0$, it is obvious that

$$\|II_2^2\|_{L^2} \leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{L^2}.$$

Similarly, we have

$$\|II_2^3\|_{H^s} \leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s} (s > 0),$$

$$\|II_2^3\|_{L^2} \leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{L^2}.$$

This proves that

$$\|T_V \cdot T_p \nabla u - T_p T_V \cdot \nabla u\|_{H^s} \leq C \|p\|_{L^\infty} \|V\|_{W^{1,\infty}} \|u\|_{H^s} (s > 0),$$

$$\|T_V \cdot T_p \nabla u - T_p T_V \cdot \nabla u\|_{L^2} \leq C \|p\|_{L^\infty} \|V\|_{B_{\infty,1}^1} \|u\|_{L^2},$$

which together with (2.7)-(2.9) give the proposition. \square

Remark 2.22. If the symbol $p(t, x, \xi)$ satisfies

$$M_{-\mu}^m(\partial_t + T_V \cdot \nabla p) \triangleq \sup_{|\alpha| \leq 3d/2+1} \sup_{|\xi| \geq 1/2} \|(1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha (\partial_t + T_V \cdot \nabla p)\|_{C^{-\mu}} < \infty$$

for some $\mu > 0$, then there holds

$$\begin{aligned} & \| [T_p, \partial_t + T_V \cdot \nabla] u(t) \|_{H^s} \\ & \leq C M_0^m(p) \|V(t)\|_{B_{\infty,1}^1} \|u(t)\|_{H^{s+m}} + C M_{-\mu}^m((\partial_t + T_V \cdot \nabla)p) \|u(t)\|_{H^{s+m+\mu}}, \end{aligned}$$

which can be seen from (2.7).

The following proposition will be used in the proof of well-posedness. Although the estimate is very rough, it is sufficient for our application.

Proposition 2.23. Assume that $V \in C([0, T]; W^{2,\infty})$, $\partial_t V \in C([0, T]; W^{1,\infty})$, and the symbol $p = p(t, x, \xi)$ is homogenous in ξ of order $m \in \mathbf{R}$. Then it holds that

$$\begin{aligned} & \|(\partial_t + T_V \cdot \nabla)[T_p, \partial_t + T_V \cdot \nabla]u(t)\|_{H^s} \leq C(1 + \|V(t)\|_{W^{2,\infty}} + \|\partial_t V(t)\|_{W^{1,\infty}})^2 \\ & \quad \times (M_2^m(p) + M_1^m(\partial_t p) + M_0^m(\partial_t^2 p)) \|(\partial_t + T_V \cdot \nabla)u(t), u(t)\|_{H^{m+s}}. \end{aligned}$$

Proof. Note that

$$[T_p, \partial_t + T_V \cdot \nabla]u(t) = -(T_{\partial_t p} + T_V T_{\nabla p})u - (T_V T_p - T_p T_V) \nabla u.$$

A direct calculation gives

$$\begin{aligned} & (\partial_t + T_V \cdot \nabla)[T_p, \partial_t + T_V \cdot \nabla]u(t) \\ & = -(\partial_t + T_V \cdot \nabla)((T_{\partial_t p} + T_V T_{\nabla p})u + (T_V T_p - T_p T_V) \nabla u) \\ & = -[\partial_t + T_V \cdot \nabla, T_{\partial_t p} + T_V T_{\nabla p}]u + (T_{\partial_t p} + T_V T_{\nabla p})(\partial_t + T_V \cdot \nabla)u \\ & \quad - [\partial_t + T_V \cdot \nabla, (T_V T_p - T_p T_V)](\nabla u) - [T_p, T_V](\partial_t + T_V \cdot \nabla)(\nabla u). \end{aligned}$$

It follows from Lemma 2.10 that

$$\begin{aligned} & \| (T_{\partial_t p} + T_V T_{\nabla p})(\partial_t + T_V \cdot \nabla)u \|_{H^s} \\ & \leq C (M_0^m(\partial_t p) + \|V(t)\|_{L^\infty} M_1^m(p)) \|(\partial_t + T_V \cdot \nabla)u\|_{H^{m+s}}. \end{aligned}$$

We decompose

$$\begin{aligned} & [\partial_t + T_V \cdot \nabla, T_{\partial_t p} + T_V T_{\nabla p}]u \\ & = [\partial_t + T_V \cdot \nabla, T_{\partial_t p}]u + [\partial_t + T_V \cdot \nabla, T_V]T_{\nabla p}u + T_V[\partial_t + T_V \cdot \nabla, T_{\nabla p}]u. \end{aligned}$$

Then it follows from Proposition 2.21 and Proposition 2.6 that

$$\begin{aligned} & \| [\partial_t + T_V \cdot \nabla, T_{\partial_t p} + T_V T_{\nabla p}]u \|_{H^s} \\ & \leq C (1 + \|V(t)\|_{C^{1+\varepsilon}} + \|\partial_t V\|_{L^\infty})^2 (M_0^m(\nabla_{t,x} p) + M_0^m(\nabla_{t,x}^2 p)) \|u(t)\|_{H^{m+s}}. \end{aligned}$$

We decompose

$$[T_p, T_V](\partial_t + T_V \cdot \nabla)(\nabla u) = [T_p, T_V]\nabla(\partial_t + T_V \cdot \nabla)u - [T_p, T_V]T_{\nabla V} \cdot \nabla u,$$

which along with Proposition 2.6 gives

$$\begin{aligned} & \| [T_p, T_V](\partial_t + T_V \cdot \nabla)(\nabla u) \|_{H^s} \\ & \leq C \|V(t)\|_{W^{1,\infty}} M_1^m(p) (\|(\partial_t + T_V \cdot \nabla)u\|_{H^{m+s}} + \|\nabla V(t)\|_{L^\infty} \|u\|_{H^{m+s}}). \end{aligned}$$

We write

$$[\partial_t, (T_V T_p - T_p T_V)](\nabla u) = [T_{\partial_t V}, T_p](\nabla u) + [T_V, T_{\partial_t p}](\nabla u),$$

from which and Proposition 2.6, we deduce that

$$\|[\partial_t, (T_V T_p - T_p T_V)](\nabla u)\|_{H^s} \leq C \|(V, \partial_t V)\|_{W^{1,\infty}} (M_1^m(p) + M_1^m(\partial_t p)) \|u\|_{H^{m+s}}.$$

Finally, we deduce from Corollary 2.9 that

$$\| [T_V \cdot \nabla, [T_V, T_p]](\nabla u) \|_{H^s} \leq C \|V\|_{W^{2,\infty}} M_2^m(p) \|u\|_{H^{m+s}}.$$

Putting the above estimates together, the proposition is proved. \square

3. PARABOLIC EVOLUTION EQUATION

Let $I = [z_0, z_1]$ be a finite interval. We denote by $\Gamma_\rho^m(I \times \mathbf{R}^d)$ the space of symbols $a(z; x, \xi)$ satisfying

$$\mathcal{M}_\rho^m(a) \stackrel{\text{def}}{=} \sup_{z \in I} \sup_{|\alpha| \leq 3d/2+1+\rho} \sup_{|\xi| \geq 1/2} \|(1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(z; \cdot, \xi)\|_{W^{\rho,\infty}} < +\infty.$$

We consider the parabolic evolution equation

$$\begin{cases} \partial_z w + T_a w = f, \\ w|_{z=z_0} = w_0, \end{cases} \quad (3.1)$$

where the symbol $a \in \Gamma_\rho^1(I \times \mathbf{R}^d)$ is elliptic in the sense that there exists $c_1 > 0$ such that for any $z \in I, (x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d$, it holds that

$$\operatorname{Re} a(z; x, \xi) \geq c_1 |\xi|. \quad (3.2)$$

For the elliptic estimates, we need to introduce two kinds of spaces. The first kind of spaces are intended for Sobolev elliptic estimates:

$$\begin{aligned} X^\sigma(I) & \stackrel{\text{def}}{=} \tilde{L}_z^\infty(I; H^\sigma(\mathbf{R}^d)) \cap L_z^2(I; H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)), \\ Y^\sigma(I) & \stackrel{\text{def}}{=} \tilde{L}_z^1(I; H^\sigma(\mathbf{R}^d)) + L_z^2(I; H^{\sigma-\frac{1}{2}}(\mathbf{R}^d)). \end{aligned}$$

The second kind of spaces are intended for Hölder elliptic estimates:

$$\begin{aligned} X_{p,q}^\sigma(I) &\stackrel{\text{def}}{=} \tilde{L}_z^p(I; B_{\infty,q}^{\sigma+\frac{1}{p}}), \\ Y_q^\sigma(I) &\stackrel{\text{def}}{=} \tilde{L}_z^1(I; B_{\infty,q}^\sigma) + \tilde{L}_z^2(I; B_{\infty,q}^{\sigma-\frac{1}{2}}) + \tilde{L}_z^\infty(I; B_{\infty,q}^{\sigma-1}). \end{aligned}$$

Let us recall the following parabolic estimates in Chemin-Lerner type spaces, which have been essentially proved in [32].

Proposition 3.1. *Let $a \in \Gamma_\rho^1(I \times \mathbf{R}^d)$ for some $\rho > 0$. Assume that $w_0 \in H^\sigma$ and $f \in Y^\sigma(I)$ for $\sigma \in \mathbf{R}$. Then there exists a unique solution $w \in X^\sigma(I)$ of (3.1) satisfying*

$$\|w\|_{X^\sigma(I)} \leq C(\mathcal{M}_\rho^1(a)) (\|w_0\|_{H^\sigma} + \|f\|_{Y^\sigma(I)}),$$

where C is an increasing function depending on c_1 and $|I|$.

Proposition 3.2. *Let $\sigma \in \mathbf{R}, p, q \in [1, \infty]$ and $a \in \Gamma_\rho^1(I \times \mathbf{R}^d)$ for some $\rho > 0$. Assume that $w \in X_{p,q}^\sigma(I)$ is a solution of (3.1). Then for any $\delta > 0$,*

$$\|w\|_{X_{p,q}^\sigma(I)} \leq C(\mathcal{M}_\rho^1(a)) (\|w_0\|_{B_{\infty,q}^\sigma} + \|f\|_{Y_{p,q}^\sigma(I)} + \|w\|_{\tilde{L}_z^p(I; C^{-\delta})}),$$

where $Y_{p,q}^\sigma(I) = \tilde{L}_z^p(I; B_{\infty,q}^{\sigma-1+\frac{1}{p}})$ and C is an increasing function depending on c_1 and $|I|$.

4. ELLIPTIC ESTIMATES IN A STRIP

The goal of this section is to establish the elliptic estimates in Sobolev spaces and Besov spaces. These estimates will be used to estimate the Dirichlet-Neumann operator, the velocity and the pressure.

Throughout this section, we assume that $\eta \in C^{\frac{3}{2}+\varepsilon}(\mathbf{R}^d)$ for some $\varepsilon > 0$ and there exists some $h_0 > 0$ such that

$$1 + \eta(x) \geq h_0 \quad \text{for } x \in \mathbf{R}^d. \quad (4.1)$$

Let $\mathcal{S} \triangleq \{(x, y) : x \in \mathbf{R}^d, -1 < y < \eta(x)\}$ be a strip.

In the sequel, we denote by $K_\eta = C(\|\eta\|_{C^{\frac{3}{2}+\varepsilon}})$ an increasing function depending on h_0 , which may change from line to line; $\nabla = (\partial_{x_1}, \dots, \partial_{x_d})$, $D = \frac{\nabla}{i}$ and $\Delta = \Delta_x$.

4.1. Elliptic boundary problem. We consider the elliptic equations in \mathcal{S} :

$$\Delta_{x,y} \phi = g \quad \text{in } \mathcal{S}, \quad (4.2)$$

with the Dirichlet boundary condition

$$\phi|_{y=\eta} = f, \quad \phi|_{y=-1} = f_b, \quad (4.3)$$

or the mixed boundary condition

$$\phi|_{y=\eta} = f, \quad \partial_y \phi|_{y=-1} = 0. \quad (4.4)$$

Given $f, f_b \in H^{\frac{1}{2}}(\mathbf{R}^d)$ and $g \in H^{-1}(\mathcal{S})$, the existence and uniqueness of weak solution for the elliptic equation (4.2)-(4.3) or (4.2)-(4.4) can be deduced by using Riesz theorem(see [3] for example). Moreover, the solution $\phi \in H^1(\mathcal{S})$ satisfies

$$\|\phi\|_{H^1(\mathcal{S})} \leq C(\|\eta\|_{W^{1,\infty}}, h_0) (\|(f, f_b)\|_{H^{\frac{1}{2}}} + \|g\|_{H^{-1}(\mathcal{S})}). \quad (4.5)$$

As a warm-up, we first establish the elliptic estimates in the flat strip $\mathcal{S} = [-1, 0] \times \mathbf{R}^d$ by using the parabolic estimates. The same ideas will be used to deal with the general case in the next subsections. In this subsection, we denote that $I = [-1, 0]$.

Proposition 4.1. *Let ϕ be a solution of (4.2)-(4.3). Then for any $\sigma \geq -\frac{1}{2}$, there holds*

$$\begin{aligned} \|\nabla_{x,y}\phi\|_{X^\sigma(I)} &\leq C(\|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})} + \|(f, f_b)\|_{H^{\sigma+1}} + \|g\|_{Y^\sigma(I)}), \\ \|\nabla_{x,y}\phi\|_{H^{\sigma+\frac{1}{2}}(\mathcal{S})} &\leq C(\|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})} + \|(f, f_b)\|_{H^{\sigma+1}} + \|g\|_{H^{\sigma-\frac{1}{2}}(\mathcal{S})}). \end{aligned}$$

Proof. We split the proof into three steps.

Step 1. Estimates on $[-1/2, 0]$

Let $w = \chi(y)(\partial_y - |D|)\phi$, where $\chi(y)$ is a smooth function satisfying $\chi(-1) = 0$ and $\chi(y) = 1$ for $y \in [-\frac{1}{2}, 0]$. Then w satisfies

$$(\partial_y + |D|)w = \chi(y)g + \chi'(y)(\partial_y + |D|)\phi \triangleq g_1, \quad w(-1) = 0.$$

Then for any $y \in [-1, 0]$, we have

$$w(x, y) = \int_{-1}^y e^{(y'-y)|D|} g_1(x, y') dy'.$$

Using the fact(see [40] for example) that there exists $c > 0$ such that for any $y \leq 0$ and $p \in [1, \infty]$, there holds

$$\|e^{y|D|}\Delta_j f\|_{L^p} \leq C e^{cy^{2j}} \|\Delta_j f\|_{L^p}, \quad (4.6)$$

we can easily deduce that for $\sigma_1 = \min(\sigma, \frac{1}{2})$,

$$\|w\|_{X^{\sigma_1}(I)} \leq C(\|g\|_{Y^\sigma(I)} + \|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})}).$$

On the other hand, we have

$$(\partial_y - |D|)\phi = w(y) \quad \text{for } y \in [-\frac{1}{2}, 0], \quad \phi(0) = f.$$

Hence, for $y \in [-\frac{1}{2}, 0]$,

$$\phi(x, y) = e^{y|D|}f + \int_0^y e^{-(y'-y)|D|} w(y') dy',$$

from which and (4.6), we infer that

$$\begin{aligned} \|\nabla_{x,y}\phi\|_{X^{\sigma_1}([-1/2, 0])} &\leq C(\|f\|_{H^{\sigma_1+1}} + \|w\|_{X^{\sigma_1}(I)}) \\ &\leq C(\|f\|_{H^{\sigma_1+1}} + \|g\|_{Y^\sigma(I)} + \|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})}). \end{aligned} \quad (4.7)$$

Step 2. Estimates on $[-1, -1/2]$

If we let $w = \chi(y)(\partial_y + |D|)\phi$, where $\chi(y)$ is a smooth function satisfying $\chi(0) = 1$ and $\chi(y) = 1$ for $y \in [-1, -\frac{1}{2}]$. Then we have

$$\begin{aligned} (\partial_y - |D|)w &= \chi(y)g + \chi'(y)(\partial_y - |D|)\phi, \quad w(0) = 0, \\ (\partial_y + |D|)\phi &= w \quad \text{for } y \in [-1, -\frac{1}{2}], \quad \phi(-1) = f_b. \end{aligned}$$

Then a similar argument leading to (4.7) gives

$$\|\nabla_{x,y}\phi\|_{X^{\sigma_1}([-1, -1/2])} \leq C(\|f_b\|_{H^{\sigma_1+1}} + \|g\|_{Y^\sigma(I)} + \|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})}),$$

which along with (4.7) gives

$$\|\nabla_{x,y}\phi\|_{X^{\sigma_1}(I)} \leq C(\|(f, f_b)\|_{H^{\sigma_1+1}} + \|g\|_{Y^\sigma(I)} + \|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})}). \quad (4.8)$$

A bootstrap argument will ensure that the inequality (4.8) holds for $\sigma_1 = \sigma$.

Step 3. Proof of the second result

To show the second inequality, it suffices to consider the estimate of the normal derivative. Let $\sigma + \frac{1}{2} = k + \sigma_1$ for $k \in \mathbf{N}$ and $\sigma_1 \in [0, 1)$. Using the interpolation inequality and the elliptic equation, we deduce that

$$\begin{aligned} \|\nabla^k \partial_y \phi\|_{H^{\sigma_1}(S)} &\leq C(\|\partial_y \phi\|_{L^2(I; H^{\sigma+\frac{1}{2}})} + \|\partial_y^2 \phi\|_{L^2(I; H^{\sigma-\frac{1}{2}})}) \\ &\leq C(\|\nabla_{x,y} \phi\|_{L^2(I; H^{\sigma+\frac{1}{2}})} + \|g\|_{H^{\sigma-\frac{1}{2}}(S)}) \\ &\leq C(\|(f, f_b)\|_{H^{\sigma_1+1}} + \|g\|_{H^{\sigma-\frac{1}{2}}(S)} + \|\nabla_{x,y} \phi\|_{L^2(S)}). \end{aligned}$$

Let us assume that for $\ell \in [1, k]$, there holds

$$\|\nabla^{k-\ell+1} \partial_y^\ell \phi\|_{H^{\sigma_1}(S)} \leq C(\|(f, f_b)\|_{H^{\sigma_1+1}} + \|g\|_{H^{\sigma-\frac{1}{2}}(S)} + \|\nabla_{x,y} \phi\|_{L^2(S)}).$$

Then the estimate $\|\partial_y^{k+1} \phi\|_{H^{\sigma_1}(S)}$ can be deduced from

$$\begin{aligned} \|\partial_y^{k+1} \phi\|_{H^{\sigma_1}(S)} &\leq C(\|\partial_y^{k-1} \Delta \phi\|_{L^2(I; H^{\sigma_1})} + \|g\|_{H^{\sigma-\frac{1}{2}}(S)}) \\ &\leq C(\|\partial_y \phi\|_{L^2(I; H^{\sigma+\frac{1}{2}})} + \|\partial_y^k \phi\|_{L^2(I; H^{\sigma_1+1})} + \|g\|_{H^{\sigma-\frac{1}{2}}(S)}), \end{aligned}$$

and an induction assumption. The proof is finished. \square

Next, we give the estimates of ϕ in the Besov space.

Proposition 4.2. *Let ϕ be a solution of (4.2)-(4.3) with $g = g_1 + \partial_y g_2$. Let $\sigma \in \mathbf{R}$, $q, r \in [1, \infty]$. Then it holds that for any $\delta > 0$,*

$$\|\nabla_{x,y} \phi\|_{X_{r,q}^\sigma(I)} \leq C(\|(f, f_b)\|_{B_{\infty,q}^{\sigma+1}} + \|g_1\|_{Y_q^\sigma(I)} + \|g_2\|_{X_{r,q}^\sigma(I)} + \|\nabla_{x,y} \phi\|_{L_z^2(I; C^{-\delta})}).$$

Proof. Let $w = \chi(y)(\partial_y - |D|)\phi$, where $\chi(y)$ is a smooth function satisfying $\chi(-1) = 0$ and $\chi(y) = 1$ for $y \in [-\frac{1}{2}, 0]$. Then w satisfies

$$(\partial_y + |D|)(w - \chi(y)g_2) = \chi(y)g_1 + \chi'(y)(\partial_y + |D|)\phi - \chi(y)|D|g_2 - \partial_y \chi g_2 \triangleq g$$

with $w(-1) = 0$. Then for any $y \in [-1, 0]$, we have

$$(w - \chi(y)g_2)(x, y) = \int_{-1}^y e^{(y'-y)|D|} g(y') dy'.$$

from which and (4.6), we infer that for $\sigma_1 = \min(-\delta + \frac{1}{3}, \sigma)$,

$$\|w\|_{X_{r,q}^{\sigma_1}(I)} \leq C(\|g_1\|_{Y_q^\sigma(I)} + \|g_2\|_{X_{r,q}^\sigma(I)} + \|\nabla_{x,y} \phi\|_{L_z^2(I; C^{-\delta})}).$$

On the other hand, for $y \in [-\frac{1}{2}, 0]$,

$$\phi(x, y) = e^{y|D|} f + \int_0^y e^{-(y'-y)|D|} w(y') dy',$$

from which and (4.6), we infer that

$$\begin{aligned} \|\nabla_{x,y} \phi\|_{X_{r,q}^{\sigma_1}([-\frac{1}{2}, 0])} &\leq C(\|f\|_{B_{\infty,q}^{\sigma+1}} + \|w\|_{X_{r,q}^{\sigma_1}(I)}) \\ &\leq C(\|f\|_{B_{\infty,q}^{\sigma+1}} + \|g_1\|_{Y_q^\sigma(I)} + \|g_2\|_{X_{r,q}^\sigma(I)} + \|\nabla_{x,y} \phi\|_{L_z^2(I; C^{-\delta})}). \end{aligned}$$

Similarly, we can deduce that

$$\|\nabla_{x,y} \phi\|_{X_{r,q}^{\sigma_1}([-1, -\frac{1}{2}])} \leq C(\|f\|_{B_{\infty,q}^{\sigma+1}} + \|g_1\|_{Y_q^\sigma(I)} + \|g_2\|_{X_{r,q}^\sigma(I)} + \|\nabla_{x,y} \phi\|_{L_z^2(I; C^{-\delta})}).$$

Then the proposition follows by using a bootstrap argument. \square

4.2. Flatten the boundary and parilinearization. Motivated by [23, 3], we flatten the boundary of \mathcal{S} by a regularized mapping:

$$(x, z) \in \mathbf{R}^d \times I \longmapsto (x, \rho_\delta(x, z)) \in \mathcal{S},$$

where $I = [-1, 0]$ and ρ_δ with $\delta > 0$ is given by

$$\rho_\delta(x, z) = z + (1 + z)e^{\delta z|D|}\eta(x). \quad (4.9)$$

In the sequel, we denote $\overline{\mathcal{S}} = \mathbf{R}^d \times I$ with $I = [-1, 0]$. For a function $f(x, y)$ defined in \mathcal{S} , we denote $\tilde{f}(x, z) = f(x, \rho_\delta(x, z))$.

For any $z \in [-1, 0]$, we have

$$\partial_z \rho_\delta = 1 + \eta(x) + (e^{\delta z|D|} - 1)\eta(x) + (1 + z)\delta e^{\delta z|D|}|D|\eta(x).$$

It is easy to show that

$$\begin{aligned} & \| (e^{\delta z|D|} - 1)\eta + \delta(1 + z)e^{\delta z|D|}|D|\eta \|_{L^\infty} \\ & \leq \delta \int_z^0 \| e^{\delta z'|D|}|D|\eta \|_{L^\infty} dz' + \delta \| e^{\delta z|D|}|D|\eta(x) \|_{L^\infty} \\ & \leq C\delta \| |D|\eta \|_{L^\infty} \leq C\delta \| \eta \|_{C^{1+\varepsilon}}. \end{aligned}$$

Hence, we can take δ small enough depending only on $\| \eta \|_{C^{1+\varepsilon}}$ and h_0 so that

$$\partial_z \rho_\delta(x, z) \geq \frac{h_0}{2} \quad \text{for } (x, z) \in \overline{\mathcal{S}}. \quad (4.10)$$

We have the following regularity information for the regularized map ρ_δ , which can be easily verified by the definition of ρ_δ .

Lemma 4.3. *For any $\sigma \geq 0$, there holds*

$$\| (\nabla \rho_\delta, \partial_z \rho_\delta - 1) \|_{X^{\sigma-\frac{1}{2}}(I)} + \| (\nabla \rho_\delta, \partial_z \rho_\delta - 1) \|_{H^\sigma(\overline{\mathcal{S}})} \leq K_\eta \| \eta \|_{H^{\sigma+\frac{1}{2}}}.$$

Let $p, q \in [1, \infty]$ and $\sigma \in \mathbf{R}$. Then we have

$$\| (\nabla \rho_\delta, \partial_z \rho_\delta - 1) \|_{\tilde{L}_z^p(I; B_{\infty, q}^\sigma)} \leq K_\eta \| \eta \|_{B_{\infty, q}^{\sigma+1-\frac{1}{p}}}.$$

We set $v(x, z) = \phi(x, \rho_\delta(x, z))$. It is easy to find that v satisfies

$$\partial_z^2 v + \alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v = F_0, \quad (4.11)$$

where $F_0 = \alpha g(x, \rho_\delta(x, z)) \triangleq F_{00} + \nabla_z G_0$ and the coefficients α, β, γ are defined by

$$\alpha = \frac{(\partial_z \rho_\delta)^2}{1 + |\nabla \rho_\delta|^2}, \quad \beta = -2 \frac{\partial_z \rho_\delta \nabla \rho_\delta}{1 + |\nabla \rho_\delta|^2}, \quad \gamma = \frac{1}{\partial_z \rho_\delta} (\partial_z^2 \rho_\delta + \alpha \Delta \rho_\delta + \beta \cdot \nabla \partial_z \rho_\delta). \quad (4.12)$$

We collect the following Sobolev and Hölder regularity information for the elliptic coefficients, which can be proved by using Bony's decomposition (2.3), Lemma 2.16-Lemma 2.18 and Lemma 4.3 (see Lemma 4.4 in [32]).

Lemma 4.4. *If $\sigma > \frac{3}{2}$, then it holds that*

$$\begin{aligned} & \| \alpha - 1 \|_{X^{\sigma-\frac{1}{2}}(I)} + \| \beta \|_{X^{\sigma-\frac{1}{2}}(I)} + \| \gamma \|_{X^{\sigma-\frac{3}{2}}(I)} \leq K_\eta \| \eta \|_{H^{\sigma+\frac{1}{2}}}, \\ & \| \alpha - 1 \|_{\tilde{L}_z^1(I; H^{\sigma+\frac{1}{2}})} + \| \beta \|_{\tilde{L}_z^1(I; H^{\sigma+\frac{1}{2}})} + \| \gamma \|_{\tilde{L}_z^1(I; H^{\sigma-\frac{1}{2}})} \leq K_\eta \| \eta \|_{H^{\sigma+\frac{1}{2}}}, \\ & \| \alpha \|_{\tilde{L}_z^\infty(I; C^{\frac{1}{2}+\varepsilon})} + \| \beta \|_{\tilde{L}_z^\infty(I; C^{\frac{1}{2}+\varepsilon})} + \| \gamma \|_{\tilde{L}_z^\infty(I; C^{-\frac{1}{2}+\varepsilon})} \leq K_\eta, \\ & \| \alpha \|_{\tilde{L}_z^2(I; C^{1+\varepsilon})} + \| \beta \|_{\tilde{L}_z^2(I; C^{1+\varepsilon})} + \| \gamma \|_{\tilde{L}_z^2(I; C^\varepsilon)} \leq K_\eta. \end{aligned}$$

Lemma 4.5. *It holds that*

$$\begin{aligned} \|(\alpha - 1, \beta)\|_{H^\sigma(\overline{\mathcal{S}})} &\leq K_\eta \|\eta\|_{H^{\sigma+\frac{1}{2}}} \quad \text{for } \sigma \geq 0, \\ \|\gamma\|_{H^{\sigma-1}(\overline{\mathcal{S}})} &\leq K_\eta \|\eta\|_{H^{\sigma+\frac{1}{2}}} \quad \text{for } \sigma \geq 1. \end{aligned}$$

Proof. The lemma follows from Lemma 2.13 and Lemma 4.3. \square

In order to obtain the tame elliptic estimates, we parilinearize the elliptic equation (4.11) as

$$\partial_z^2 v + T_\alpha \Delta v + T_\beta \cdot \nabla \partial_z v = F_0 + F_1 + F_2, \quad (4.13)$$

with F_1, F_2 given by

$$F_1 = \gamma \partial_z v, \quad F_2 = (T_\alpha - \alpha) \Delta v + (T_\beta - \beta) \cdot \nabla \partial_z v.$$

Following [3], we decouple the equation (4.13) into a forward and a backward parabolic evolution equations:

$$(\partial_z - T_a)(\partial_z - T_A)v = F_0 + F_1 + F_2 + F_3 \triangleq F, \quad (4.14)$$

where

$$\begin{aligned} a &= \frac{1}{2}(-i\beta \cdot \xi - \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}), \\ A &= \frac{1}{2}(-i\beta \cdot \xi + \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}), \\ F_3 &= (T_a T_A - T_\alpha \Delta)v - (T_a + T_A + T_\beta \cdot \nabla) \partial_z v - T_{\partial_z A} v. \end{aligned}$$

The symbols a, A satisfy

Lemma 4.6. *It holds that*

$$\begin{aligned} a(z; x, \xi) \cdot A(z; x, \xi) &= -\alpha(x, z)|\xi|^2, \\ a(z; x, \xi) + A(z; x, \xi) &= -i\beta(x, z) \cdot \xi. \end{aligned}$$

Moreover, $a, A \in \mathcal{M}_{\frac{1}{2}+\varepsilon}^1$ with the bounds

$$\mathcal{M}_{\frac{1}{2}+\varepsilon}^1(a) \leq K_\eta, \quad \mathcal{M}_{\frac{1}{2}+\varepsilon}^1(A) \leq K_\eta.$$

Proof. The two equalities are obvious. Note that

$$4\alpha|\xi|^2 - (\beta \cdot \xi)^2 \geq c_2|\xi|^2$$

for some $c_2 > 0$ depending only on $\|\eta\|_{C^{\frac{3}{2}+\varepsilon}}$. Then the second statement follows from Lemma 2.12 and Lemma 4.4. \square

4.3. Elliptic estimates in Sobolev space. We first present the trace estimate in terms of the H^1 norm of the solution.

Lemma 4.7. *Assume that $v \in H^1(\overline{\mathcal{S}})$ is a solution of (4.11) in $\overline{\mathcal{S}}$. Then we have*

$$\begin{aligned} \|\nabla v\|_{X^{-\frac{1}{2}}(I)} &\leq C\|v\|_{H^1(\overline{\mathcal{S}})}, \\ \|\partial_z v\|_{X^{-\frac{1}{2}}(I)} &\leq K_\eta(\|F_0\|_{Y^{-\frac{1}{2}}(I)} + \|v\|_{H^1(\overline{\mathcal{S}})}). \end{aligned}$$

Proof. Let $v_j = \Delta_j v$. Let $\chi(z)$ be a smooth function supported in $[-\frac{3}{4}, 0]$ and $\chi(z) = 1$ on $[-\frac{1}{2}, 0]$. For any $z \in [-\frac{1}{2}, 0]$, we have

$$\begin{aligned} \langle \nabla v_j(z), \nabla v_j(z) \rangle_{H^{-\frac{1}{2}}} &\leq 2 \int_{-1}^z \chi(z') \langle \partial_{z'} \nabla v_j(z'), \nabla v_j(z') \rangle_{H^{-\frac{1}{2}}} dz' \\ &\quad + \int_{-1}^z \chi'(z') \langle \nabla v_j(z'), \nabla v_j(z') \rangle_{H^{-\frac{1}{2}}} dz' \\ &\leq C \|\nabla_{x,z} v_j\|_{L^2(\overline{\mathcal{S}})}^2 + C \|v\|_{L^2(\overline{\mathcal{S}})}^2 \end{aligned}$$

which implies that

$$\|\nabla v\|_{\tilde{L}^\infty(-\frac{1}{2}, 0; H^{-\frac{1}{2}})} \leq C \|v\|_{H^1(\overline{\mathcal{S}})}.$$

By the equation (4.11) and Lemma 4.8 in [32], we get

$$\begin{aligned} \langle \partial_z v_j(z), \partial_z v_j(z) \rangle_{H^{-\frac{1}{2}}} &= 2 \int_{-1}^z \chi(z') \langle \partial_{z'}^2 v_j(z'), \partial_z v_j(z') \rangle_{H^{-\frac{1}{2}}} dz' \\ &\quad + \int_{-1}^z \chi'(z') \langle \partial_{z'} v_j(z'), \partial_z v_j(z') \rangle_{H^{-\frac{1}{2}}} dz' \\ &= 2 \int_{-1}^z \chi(z') \langle \Delta_j (F_0 - \alpha \Delta v + \beta \nabla \partial_z v - \gamma \partial_z v), \partial_z v_j \rangle_{H^{-\frac{1}{2}}} dz' \\ &\quad + \int_{-1}^z \chi'(z') \langle \partial_{z'} v_j(z'), \partial_z v_j(z') \rangle_{H^{-\frac{1}{2}}} dz' \\ &\leq C c_j \left(\|F_0\|_{Y^{-\frac{1}{2}}(I)} + \|\operatorname{div}_x(\alpha \nabla v + \beta \partial_z v)\|_{L_z^2(I; H^{-1})} \right. \\ &\quad \left. + \|\nabla \alpha \nabla v + \nabla \beta \partial_z v + \gamma \partial_z v\|_{\tilde{L}_z^1(I; H^{-\frac{1}{2}})} + \|\nabla_{x,z} v\|_{L^2(I \times \mathbf{R}^d)} \right) \|\partial_z v\|_{X^{-\frac{1}{2}}(I)} \\ &\leq K_\eta c_j (\|F_0\|_{Y^{-\frac{1}{2}}(I)} + \|v\|_{H^1(\overline{\mathcal{S}})}) \|\partial_z v\|_{X^{-\frac{1}{2}}(I)}, \end{aligned}$$

with $\|\{c_j\}\|_{\ell^1} \leq 1$, which implies that

$$\|\partial_z v\|_{\tilde{L}^\infty(-\frac{1}{2}, 0; H^{-\frac{1}{2}})}^2 \leq K_\eta (\|F_0\|_{Y^{-\frac{1}{2}}(I)} + \|\nabla_{x,z} v\|_{L^2(\overline{\mathcal{S}})}) \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}(I)}.$$

The same estimates hold for $z \in [-1, -\frac{1}{2}]$ by a similar cut-off argument. This gives

$$\|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}(I)}^2 \leq K_\eta (\|F_0\|_{Y^{-\frac{1}{2}}(I)} + \|\nabla_{x,z} v\|_{L^2(\overline{\mathcal{S}})}) \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}(I)},$$

which implies the desired result. \square

Remark 4.8. If $\eta \in H^s(\mathbf{R}^d)$ for $s > \frac{d}{2} + 1$, then we have

$$\|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}(I)} \leq C(\|\eta\|_{H^s}, h_0) (\|F_0\|_{Y^{-\frac{1}{2}}(I)} + \|v\|_{H^1(\overline{\mathcal{S}})}).$$

Indeed, we have $\nabla \alpha, \nabla \beta, \gamma \in H^{s-\frac{3}{2}}(\overline{\mathcal{S}})$, which implies by Lemma 2.10 that

$$\|\nabla \alpha \nabla v + \nabla \beta \partial_z v + \gamma \partial_z v\|_{L_z^1(I; H^{-\frac{1}{2}})} \leq C(\|\eta\|_{H^s}, h_0) \|\nabla_{x,z} v\|_{L^2(\overline{\mathcal{S}})}.$$

The following elliptic estimates will be used to estimate the velocity in the proof of well-posedness. Here and in what follows, we denote by $P_{\sigma, \eta} = C(\|\eta\|_{H^\sigma})$ an increasing function depending on h_0 , which may be different from line to line.

Let us first recall the following elliptic estimate for tangential derivatives, which has been essentially proved in [3].

Proposition 4.9. *Let $v \in H^1(\overline{\mathcal{S}})$ be a solution of (4.11) in $\overline{\mathcal{S}}$ with $v(x, 0) = f(x)$ and $v(x, -1) = f_b(x)$. Assume that $\eta \in H^s(\mathbf{R}^d)$ for $s > \frac{d}{2} + \frac{3}{2}$. Then for any $\sigma \in [-\frac{1}{2}, s - \frac{1}{2}]$, it holds that*

$$\begin{aligned} \|\nabla_{x,z} v\|_{X^{\sigma-\frac{1}{2}}(I)} &\leq P_{s,\eta}(\|\nabla_{x,z} v\|_{L^2(\overline{\mathcal{S}})} + \|(f, f_b)\|_{H^{\sigma+\frac{1}{2}}} + \|F_0\|_{L_z^2(I; H^{\sigma-1})}), \\ \|\nabla_{x,z} v\|_{X^{\sigma-\frac{1}{2}}([a,0])} &\leq P_{s,\eta}(\|\nabla_{x,z} v\|_{L^2(\overline{\mathcal{S}})} + \|f\|_{H^{\sigma+\frac{1}{2}}} + \|F_0\|_{L_z^2(I; H^{\sigma-1})}), \end{aligned}$$

for any $a \in (-1, 0)$.

Now we present the elliptic estimate for full derivatives.

Proposition 4.10. *Let $v \in H^1(\overline{\mathcal{S}})$ be a solution of (4.11) in $\overline{\mathcal{S}}$ with $v(x, 0) = f(x)$ and $v(x, -1) = f_b(x)$. Assume that $\eta \in H^s(\mathbf{R}^d)$ for $s > \frac{d}{2} + \frac{3}{2}$. Then it holds that*

$$\|\nabla_{x,z} v\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} \leq P_{s,\eta}(\|\nabla_{x,z} v\|_{L^2(\overline{\mathcal{S}})} + \|(f, f_b)\|_{H^s} + \|F_0\|_{H^{s-\frac{3}{2}}(\overline{\mathcal{S}})}).$$

We need the following lemma.

Lemma 4.11. *Let $\sigma = s - \frac{1}{2}$ and $\sigma_1 = \sigma - [\sigma]$. Then for any positive integer $k \leq [\sigma]$, there holds*

$$\begin{aligned} &\|\partial_z^{k-1}(\alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v)\|_{L_z^2(I; H^{\sigma_1})} \\ &\leq P_{s,\eta} \left(\|\langle D \rangle^{\sigma_1} \nabla v\|_{H^k(\overline{\mathcal{S}})} + \|\langle D \rangle^{\sigma_1+\frac{1}{2}} \partial_z v\|_{H^{k-1}(\overline{\mathcal{S}})} + \|\nabla_{x,z} v\|_{L^\infty(\overline{\mathcal{S}})} \right. \\ &\quad \left. + \|\langle D \rangle^{\sigma_1+\frac{1}{2}} \nabla_{x,z} v\|_{L_z^\infty(I; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \nabla_{x,z} v\|_{L_z^\infty(I; L^2)} \right). \end{aligned}$$

Proof. For $k = 1$, we have by Lemma 2.11 and Lemma 4.4 that

$$\begin{aligned} &\|\alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v\|_{L_z^2(I; H^{\sigma_1})} \\ &\leq C(\|(\alpha - 1, \beta)\|_{L^\infty} + 1) \|\langle D \rangle^{\sigma_1} \nabla v\|_{H^1(\overline{\mathcal{S}})} + \|\gamma\|_{L_z^\infty(I; C^{-\frac{1}{2}})} \|\langle D \rangle^{\sigma_1+\frac{1}{2}} \partial_z v\|_{L^2(\overline{\mathcal{S}})} \\ &\quad + C \|\nabla_{x,z} v\|_{L^\infty(\overline{\mathcal{S}})} (\|(\alpha - 1, \beta)\|_{L^2(I; H^{1+\sigma_1})} + \|\gamma\|_{L^2(I; H^{\sigma_1})}) \\ &\leq P_{s,\eta}(\|\nabla_{x,z} v\|_{L^\infty(\overline{\mathcal{S}})} + \|\langle D \rangle^{\sigma_1} \nabla v\|_{H^1(\overline{\mathcal{S}})} + \|\langle D \rangle^{\sigma_1+\frac{1}{2}} \partial_z v\|_{L^2(\overline{\mathcal{S}})}). \end{aligned}$$

Here we used $\sigma_1 + \frac{3}{2} \leq s$ and $s > \frac{d}{2} + \frac{3}{2}$.

Next we consider the case of $k \geq 2$. Let $(\overline{v}, \overline{\alpha}, \overline{\beta}, \overline{\gamma})$ be an extension of $(v, \alpha, \beta, \gamma)$ to \mathbf{R}^{d+1} so that

$$\begin{aligned} \|\nabla_{x,z} \overline{v}\|_{H^\sigma(\mathbf{R}^{d+1})} &\leq C \|\nabla_{x,z} v\|_{H^\sigma(\overline{\mathcal{S}})}, \\ \|(\overline{\alpha} - 1, \overline{\beta})\|_{H^\sigma(\mathbf{R}^{d+1})} &\leq C \|(\alpha - 1, \beta)\|_{H^\sigma(\overline{\mathcal{S}})}, \\ \|\overline{\gamma}\|_{H^{\sigma-1}(\mathbf{R}^{d+1})} &\leq C \|\gamma\|_{H^{\sigma-1}(\overline{\mathcal{S}})}. \end{aligned}$$

Using Bony's decomposition (2.3), we write

$$\begin{aligned} \overline{\beta} \nabla \partial_z \overline{v} &= T_{\overline{\beta}} \nabla \partial_z \overline{v} + T_{\nabla \partial_z \overline{v}} \overline{\beta} + R(\overline{\beta}, \nabla \partial_z \overline{v}) \\ &= T_{\overline{\beta}} \nabla \partial_z \overline{v} + \partial_z T_{\nabla \partial_z \overline{v}} \overline{\beta} - T_{\nabla \partial_z \overline{v}} \partial_z \overline{\beta} + R(\overline{\beta}, \nabla \partial_z \overline{v}), \end{aligned}$$

from which and Lemma 2.14, we infer that for any $\epsilon > 0$

$$\begin{aligned} \|\overline{\beta} \nabla \partial_z \overline{v}\|_{H^{k-1, \sigma_1}} &\leq C(\|\overline{\beta}\|_{L^\infty} + \|\langle D \rangle^{\sigma_1+\epsilon} \overline{\beta}\|_{L_z^\infty(\mathbf{R}; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \overline{\beta}\|_{L_z^\infty(\mathbf{R}; L^2)}) \|\nabla \overline{v}\|_{H^{k, \sigma_1}} \\ &\quad + C(\|\nabla_{x,z} \overline{v}\|_{L^\infty} + \|\langle D \rangle^{\sigma_1+\epsilon} \nabla \overline{v}\|_{L_z^\infty(\mathbf{R}; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \nabla \overline{v}\|_{L_z^\infty(\mathbf{R}; L^2)}) \|\overline{\beta}\|_{H^{k, \sigma_1}}. \end{aligned}$$

Due to $\sigma_1 + \epsilon + 1 \leq s$ for some $\epsilon \in (0, 1]$ and $s > \frac{d}{2} + \frac{3}{2}$, it follows from Lemma 4.5 that

$$\|\bar{\beta}\|_{L^\infty} + \|\langle D \rangle^{\sigma_1 + \epsilon} \bar{\beta}\|_{L_z^\infty(\mathbf{R}; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \bar{\beta}\|_{L_z^\infty(\mathbf{R}; L^2)} + \|\bar{\beta}\|_{H^{k, \sigma_1}} \leq P_{s, \eta}.$$

This shows that

$$\begin{aligned} \|\langle D \rangle^{\sigma_1} (\beta \nabla \partial_z v)\|_{H^{k-1}(\bar{\mathcal{S}})} &\leq P_{s, \eta} (\|\langle D \rangle^{\sigma_1} \nabla v\|_{H^k(\bar{\mathcal{S}})} + \|\nabla_{x, z} v\|_{L^\infty(\bar{\mathcal{S}})} \\ &\quad + \|\langle D \rangle^{\sigma_1 + \epsilon} \nabla v\|_{L_z^\infty(I; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \nabla v\|_{L_z^\infty(I; L^2)}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|\langle D \rangle^{\sigma_1} (\alpha \Delta v)\|_{H^{k-1}(\bar{\mathcal{S}})} &\leq P_{s, \eta} (\|\langle D \rangle^{\sigma_1} \nabla v\|_{H^k(\bar{\mathcal{S}})} + \|\nabla_{x, z} v\|_{L^\infty(\bar{\mathcal{S}})} \\ &\quad + \|\langle D \rangle^{\sigma_1 + \epsilon} \nabla v\|_{L_z^\infty(I; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \nabla v\|_{L_z^\infty(I; L^2)}). \end{aligned}$$

While, we can see from the proof of Lemma 2.14 that

$$\begin{aligned} \|\langle D \rangle^{\sigma_1} \bar{\gamma} \partial_z \bar{v}\|_{H^{k-1, \sigma_1}} &\leq C (\|\bar{\gamma}\|_{L_z^\infty(\mathbf{R}; C^{-\frac{1}{2}})} + \|\langle D \rangle^{\sigma_1 + \epsilon} \bar{\gamma}\|_{L^2} + \|\langle D \rangle^{\frac{d}{2}} \bar{\gamma}\|_{L^2}) \|\langle D \rangle^{\sigma_1 + \frac{1}{2}} \partial_z \bar{v}\|_{H^{k-1}} \\ &\quad + C (\|\nabla_{x, z} \bar{v}\|_{L^\infty} + \|\langle D \rangle^{\sigma_1 + \epsilon} \partial_z \bar{v}\|_{L_y^\infty(\mathbf{R}; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \partial_z \bar{v}\|_{L_y^\infty(\mathbf{R}; L^2)}) \|\bar{\gamma}\|_{H^{k-1, \sigma_1}}. \end{aligned}$$

We know from Lemma 4.4 and Lemma 4.5 that

$$\|\bar{\gamma}\|_{L_z^\infty(\mathbf{R}; C^{-\frac{1}{2}})} + \|\langle D \rangle^{\sigma_1 + \epsilon} \bar{\gamma}\|_{L^2} + \|\langle D \rangle^{\frac{d}{2}} \bar{\gamma}\|_{L^2} + \|\bar{\gamma}\|_{H^{k-1, \sigma_1}} \leq P_{s, \eta}.$$

Then we infer that

$$\begin{aligned} \|\gamma \partial_z v\|_{H^{k-1}(\bar{\mathcal{S}})} &\leq P_{s, \eta} (\|\langle D \rangle^{\sigma_1} \partial_z v\|_{H^{k-\frac{1}{2}}(\bar{\mathcal{S}})} + \|\nabla_{x, z} v\|_{L^\infty(\bar{\mathcal{S}})} \\ &\quad + \|\langle D \rangle^{\sigma_1 + \epsilon} \partial_z v\|_{L_z^\infty(I; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \partial_z v\|_{L_z^\infty(I; L^2)}). \end{aligned}$$

Summing up the above estimates, we conclude the lemma. \square

Proof of Proposition 4.10. Let $\sigma = s - \frac{1}{2}$. Proposition 4.9 ensures that

$$\begin{aligned} &\|\nabla_{x, z} v\|_{L_z^\infty(I; H^{\sigma - \frac{1}{2}})} + \|\nabla_{x, z} v\|_{L_z^2(I; H^\sigma)} \\ &\leq P_{s, \eta} (\|\nabla_{x, z} v\|_{L^2(\bar{\mathcal{S}})} + \|(f, f_b)\|_{H^{\sigma + \frac{1}{2}}} + \|F_0\|_{L_z^2(I; H^{\sigma-1})}). \end{aligned} \quad (4.15)$$

Assume that for $1 \leq \ell \leq [\sigma] - 1$, there holds

$$\|\nabla_{x, z}^{\ell+1} v\|_{L_z^2(I; H^{\sigma-\ell})} \leq P_{s, \eta} (\|\nabla_{x, z} v\|_{L^2(\bar{\mathcal{S}})} + \|(f, f_b)\|_{H^{\sigma + \frac{1}{2}}} + \|F_0\|_{H^{\sigma-1}(\bar{\mathcal{S}})}). \quad (4.16)$$

We will prove that the inequality holds for $\ell = [\sigma] \triangleq k$. Let $\sigma_1 = \sigma - k$. Using the equation (4.11) and Lemma 4.11, we deduce that

$$\begin{aligned} \|\partial_z^{k+1} v\|_{L_z^2(I; H^{\sigma_1})} &\leq \|\partial_z^{k-1} (\alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v)\|_{L_z^2(I; H^{\sigma_1})} + \|\partial_z^{k-1} F_0\|_{L_z^2(I; H^{\sigma_1})} \\ &\leq P_{s, \eta} (\|\langle D \rangle^{\sigma_1} \nabla v\|_{H^k(\bar{\mathcal{S}})} + \|\langle D \rangle^{\sigma_1 + \frac{1}{2}} \partial_z v\|_{H^{k-1}(\bar{\mathcal{S}})} + \|\nabla_{x, z} v\|_{L_z^\infty(I; H^{\sigma - \frac{1}{2}})} \\ &\quad + \|\langle D \rangle^{\sigma_1 + \frac{1}{2}} \nabla_{x, z} v\|_{L_z^\infty(I; L^2)} + \|\langle D \rangle^{\frac{d}{2}} \nabla_{x, z} v\|_{L_z^\infty(I; L^2)}) + \|F_0\|_{H^{\sigma-1}(\bar{\mathcal{S}})}. \end{aligned} \quad (4.17)$$

Note that $\sigma_1 + \frac{1}{2} \leq \sigma - \frac{1}{2}$, $\frac{d}{2} < \sigma - \frac{1}{2}$. This together with (4.15) and (4.16) ensures that the inequality (4.16) holds for $\ell = [\sigma]$.

Using Lemma 2.13, Lemma 4.5 and the interpolation, we obtain

$$\begin{aligned} \|\partial_z^{k+1} v\|_{H^{\sigma_1}(\bar{\mathcal{S}})} &\leq \|\partial_z^{k-1} (\alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v)\|_{H^{\sigma_1}(\bar{\mathcal{S}})} + \|\partial_z^{k-1} F_0\|_{H^{\sigma_1}(\bar{\mathcal{S}})} \\ &\leq P_{s, \eta} (\|\nabla \nabla_{x, z} v\|_{H^{k-1+\sigma_1}(\bar{\mathcal{S}})} + \|\langle D \rangle^{\frac{1}{2}} \partial_z v\|_{H^{k-1+\sigma_1}(\bar{\mathcal{S}})}) + \|F_0\|_{H^{\sigma-1}(\bar{\mathcal{S}})} \end{aligned}$$

$$\leq P_{s,\eta}(\|\nabla_{x,z}v\|_{L_z^2(I;H^{k+\sigma_1})} + \|\partial_z^{k+1}v\|_{L_z^2(I;H^{\sigma_1})}) + \|F_0\|_{H^{\sigma-1}(\bar{\mathcal{S}})},$$

which together with (4.15) and (4.17) implies the proposition. \square

4.4. Tame elliptic estimates. To prove the break-down criterion, we need to establish the tame elliptic estimates. In this subsection, we assume that $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ for $s > \frac{d}{2} + 1$.

The following elliptic estimate will be used to estimate the Dirichlet-Neumann operator and the pressure.

Proposition 4.12. *Let $I_0 = [a, 0]$ for $a \in (-1, 0)$. Assume that $v \in H^{s+1}(\bar{\mathcal{S}})$ is a solution of (4.11) in $\bar{\mathcal{S}}$ with $v(x, 0) = f(x)$. Then there exists an interval $I_1 \subset (-1, 0)$ so that for all $\sigma \in [-\frac{1}{2}, s - \frac{1}{2}]$,*

$$\|\nabla_{x,z}v\|_{X^\sigma(I_0)} \leq K_\eta(\|\nabla_{x,z}v\|_{L^2(\bar{\mathcal{S}})} + \|f\|_{H^{\sigma+1}} + \|F_{00}\|_{Y^\sigma(I)} + \|G_0\|_{X^\sigma(I)} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z}v\|_{L^\infty(\tilde{\mathcal{S}})}).$$

Here $\tilde{\mathcal{S}} = \mathbf{R}^d \times I_1$. Moreover, if $\sigma < s - \frac{1}{2}$, $\|\nabla_{x,z}v\|_{L^\infty(\tilde{\mathcal{S}})}$ can be replaced by $\|\nabla_{x,z}v\|_{L_z^\infty(I_1;C^0)}$.

The proof of the proposition need the following lemma, which can be proved by using Bony's decomposition, Lemma 2.16 and Lemma 2.18. Since the proof is almost the same as those in [32], we omit the details.

Lemma 4.13. *Let $I_1 \subseteq I$ be an interval and $\tilde{\mathcal{S}} = \mathbf{R}^d \times I_1$. For any $\sigma \leq s - \frac{1}{2}$, it holds that*

$$\begin{aligned} \|F_1\|_{Y^\sigma(I_1)} &\leq K_\eta(\|\partial_z v\|_{L_z^2(I_1;H^\sigma)} + \|\partial_z v\|_{L^\infty(\tilde{\mathcal{S}})} \|\eta\|_{H^{s+\frac{1}{2}}}), \\ \|F_2\|_{Y^\sigma(I_1)} &\leq K_\eta \|\nabla_{x,z}v\|_{L^\infty(\tilde{\mathcal{S}})} \|\eta\|_{H^{s+\frac{1}{2}}}, \\ \|F_3\|_{Y^\sigma(I_1)} &\leq K_\eta \|\nabla v\|_{L_z^2(I_1;H^\sigma)} \quad \text{for any } \sigma \in \mathbf{R}. \end{aligned}$$

If $\sigma < s - \frac{1}{2}$, $\|\nabla_{x,z}v\|_{L^\infty(\tilde{\mathcal{S}})}$ can be replaced by $\|\nabla_{x,z}v\|_{L_z^\infty(I_1;C^0)}$.

Proof of Proposition 4.12. As in [3], the proof uses the induction argument. Suppose that there exists $I_2 = [\xi_0, 0] \subseteq I_1$ with $\xi_0 \in (-1, a)$ such that for some $\sigma \in [-\frac{1}{2}, s - \frac{1}{2}]$, there holds

$$\|\nabla_{x,z}v\|_{X^\sigma(I_2)} \leq K_\eta(\|\nabla_{x,z}v\|_{L^2(\bar{\mathcal{S}})} + \|f\|_{H^{\sigma+1}} + \|F_{00}\|_{Y^\sigma(I)} + \|G_0\|_{X^\sigma(I)} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z}v\|_{L^\infty(\tilde{\mathcal{S}})}).$$

This is indeed true for $\sigma = -\frac{1}{2}$ by Lemma 4.7. We will show that the above inequality still holds when σ and I_2 are replaced by $\sigma + \delta_1$ and I_3 , where $\sigma + \delta_1 \leq s - \frac{1}{2}$ with $\delta_1 \in (0, \frac{1}{2}]$, and $I_3 = [\xi_1, 0]$ with $\xi_1 = \xi_0 + \frac{1}{2}(a - \xi_0) \in (\xi_0, a)$.

Let χ be a smooth function such that $\chi(\xi_0) = 0$ and $\chi(z) = 1$ for $z \geq \xi_1$. Set $w = \chi(z)(\partial_z - T_A)v - \chi(z)G_0$. Then w satisfies

$$\partial_z w - T_A w = F', \quad w(\xi_0) = 0,$$

where $F' = \chi(z)(F_{00} + F_1 + F_2 + F_3) + \chi'(z)((\partial_z - T_A)v - G_0) + \chi(z)T_A G_0$.

We deduce from Proposition 2.6 and Lemma 4.6 that for $\delta_1 \leq \frac{1}{2}$,

$$\|(\partial_z - T_A)v\|_{L^2(I_2;H^{\sigma+\delta_1})} \leq K_\eta \|\nabla_{x,z}v\|_{L^2(I_2;H^{\sigma+\delta_1})},$$

which along with Lemma 4.13 gives

$$\begin{aligned} \|F'\|_{Y^{\sigma+\delta_1}(I_2)} &\leq \|F_0\|_{Y^{\sigma+\delta_1}(I_2)} + K_\eta \|G_0\|_{X^\sigma(I_2)} \\ &\quad + K_\eta(\|\nabla_{x,z}v\|_{L^\infty(I_2 \times \mathbf{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\nabla_{x,z}v\|_{X^\sigma(I_2)}). \end{aligned} \quad (4.18)$$

Then it follows from Proposition 3.1 that

$$\begin{aligned} \|w\|_{X^{\sigma+\delta_1}(I_2)} &\leq K_\eta \|F\|_{Y^{\sigma+\delta_1}(I_2)} \\ &\leq K_\eta (\|F_0\|_{Y^{\sigma+\delta_1}(I)} + \|G_0\|_{X^\sigma(I)} + \|\nabla_{x,z}v\|_{X^\sigma(I_2)} + \|\nabla_{x,z}v\|_{L^\infty(I_2 \times \mathbf{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned} \quad (4.19)$$

To obtain the estimate of v , we consider the backward parabolic equation

$$\partial_z v - T_A v = w + G_0 \quad \text{in } I_3, \quad v|_{z=0} = f.$$

Take ∇ to the equation of v to get

$$(\partial_z - T_A)\nabla v = \nabla(w + G_0) + T_{\nabla A}v, \quad \nabla v|_{z=0} = \nabla f.$$

By Remark 2.8 and Lemma 4.6, we have

$$\|T_{\nabla A}v\|_{L_z^2(I_3; H^{\sigma+\delta_1-\frac{1}{2}})} \leq K_\eta \|\nabla v\|_{L_z^2(I_3; H^{\sigma+\delta_1})}.$$

Then Proposition 3.1, (4.19) and the induction assumption ensure that

$$\begin{aligned} \|\nabla v\|_{X^{\sigma+\delta_1}(I_3)} &\leq K_\eta (\|w\|_{X^{\sigma+\delta_1}(I_3)} + \|G_0\|_{X^{\sigma+\delta_1}(I_3)} + \|f\|_{H^{\sigma+1+\delta_1}} + \|\nabla v\|_{L^2(I_3; H^{\sigma+\delta_1})}) \\ &\leq K_\eta (\|f\|_{H^{\sigma+1+\delta_1}} + \|F_0\|_{Y^{\sigma+\delta_1}(I)} + \|G_0\|_{X^{\sigma+\delta_1}(I_3)} + \|\nabla_{x,z}v\|_{X^\sigma(I_2)} + \|\nabla_{x,z}v\|_{L^\infty(I_2 \times \mathbf{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}}) \\ &\leq K_\eta (\|\nabla_{x,z}v\|_{L^2(\overline{\mathcal{S}})} + \|f\|_{H^{\sigma+\delta_1+1}} + \|F_0\|_{Y^{\sigma+\delta_1}(I)} + \|G_0\|_{X^{\sigma+\delta_1}(I_3)} + \|\nabla_{x,z}v\|_{L^\infty(I_2 \times \mathbf{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

The same estimate holds for $\partial_z v$ by using $\partial_z v = T_A v + w + G_0$.

In the case when $\sigma + \delta_1 < s - \frac{1}{2}$, $\|\nabla_{x,z}v\|_{L^\infty(I_2 \times \mathbf{R}^d)}$ in (4.18) can be replaced by $\|\nabla_{x,z}v\|_{L_z^\infty(I_2; B_{\infty,\infty}^0)}$ by Lemma 4.13, and so does the final result. \square

The following elliptic estimate will be used to estimate the velocity in the proof of break-down criterion.

Proposition 4.14. *Assume that $v \in H^{s+\frac{1}{2}}(\overline{\mathcal{S}})$ is a solution of (4.11) in $\overline{\mathcal{S}}$ with $v(x, 0) = f(x)$ and $v(x, -1) = f_b(x)$. Let $k = s - \frac{1}{2}$ be an integer. Then it holds that*

$$\begin{aligned} \|\nabla_{x,z}v\|_{H^k(\overline{\mathcal{S}})} &\leq K_\eta (\|\nabla_{x,z}v\|_{L^2(\overline{\mathcal{S}})} + \|(f, f_b)\|_{H^s} \\ &\quad + \|F_0\|_{H^{k-1}(\overline{\mathcal{S}})} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z}v\|_{L^\infty(\overline{\mathcal{S}})}). \end{aligned}$$

Let us first present the tame estimate for the tangential derivatives.

Lemma 4.15. *Assume that $v \in H^{s+1}(\overline{\mathcal{S}})$ is a solution of (4.11) in $\overline{\mathcal{S}}$ with $v(x, 0) = f(x)$ and $v(x, -1) = f_b(x)$. Then for all $\sigma \in [-\frac{1}{2}, s - \frac{1}{2}]$, it holds that*

$$\|\nabla_{x,z}v\|_{X^\sigma(I)} \leq K_\eta (\|\nabla_{x,z}v\|_{L^2(\overline{\mathcal{S}})} + \|(f, f_b)\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma(I)} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z}v\|_{L^\infty(\overline{\mathcal{S}})}).$$

Proof. Because the proof is similar to Proposition 4.12, we just present a sketch. Assume that for some $\sigma \in [-\frac{1}{2}, s - \frac{1}{2}]$, there holds

$$\|\nabla_{x,z}v\|_{X^\sigma(I)} \leq K_\eta (\|\nabla_{x,z}v\|_{L^2(\overline{\mathcal{S}})} + \|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma(I)} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z}v\|_{L^\infty(\overline{\mathcal{S}})}).$$

We show that the above inequality still holds when σ is replaced by $\sigma + \delta_1$, where $\sigma + \delta_1 \leq s - \frac{1}{2}$ with $\delta_1 \in (0, \frac{1}{2}]$.

Let χ be a smooth function such that $\chi(-1) = 0$ and $\chi(z) = 1$ for $z \in [-\frac{1}{2}, 0]$. Set $w = \chi(z)(\partial_z - T_A)v$. Then (w, v) satisfies

$$\begin{aligned} \partial_z w - T_A w &= \tilde{F}, \quad w(-1) = 0, \\ \partial_z v - T_A v &= w, \quad v(0) = f, \end{aligned}$$

where $\tilde{F} = \chi(z)(F_0 + F_1 + F_2 + F_3) + \chi'(z)(\partial_z - T_A)v$. Using the same argument of Proposition 4.12, we can deduce that

$$\begin{aligned} & \|\nabla_{x,z}v\|_{X^{\sigma+\delta_1}([-\frac{1}{2},0])} \\ & \leq K_\eta(\|\nabla_{x,z}v\|_{L^2(\mathcal{S})} + \|f\|_{H^{\sigma+\delta_1+1}} + \|F_0\|_{Y^{\sigma+\delta_1}(I)} + \|\nabla_{x,z}v\|_{L^\infty(\mathcal{S})}\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

On the other hand, the equation (4.13) can also be decoupled into a backward and forward parabolic evolution equations, i.e.,

$$(\partial_z - T_A)(\partial_z - T_a)v = F_0 + F_1 + F_2 + \tilde{F}_3,$$

where

$$\tilde{F}_3 = (T_A T_a - T_a \Delta)v - (T_a + T_A + T_\beta \cdot \nabla)\partial_z v - T_{\partial_z a}v.$$

Now we let χ be a smooth function such that $\chi(0) = 0$ and $\chi(z) = 1$ for $z \in [-1, -\frac{1}{2}]$. Set $w = \chi(z)(\partial_z - T_a)v$. Then (w, v) satisfies

$$\begin{aligned} \partial_z w - T_A w &= \tilde{F}, \quad w(0) = 0, \\ \partial_z v - T_a v &= w, \quad v(-1) = f_b, \end{aligned}$$

A similar argument ensures that

$$\begin{aligned} & \|\nabla_{x,z}v\|_{X^{\sigma+\delta_1}([-1, -\frac{1}{2}])} \\ & \leq K_\eta(\|\nabla_{x,z}v\|_{L^2(\mathcal{S})} + \|f_b\|_{H^{\sigma+\delta_1+1}} + \|F_0\|_{Y^{\sigma+\delta_1}(I)} + \|\nabla_{x,z}v\|_{L^\infty(\mathcal{S})}\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

This together with the estimates in the interval $[-\frac{1}{2}, 0]$ gives the lemma. \square

Proof of Proposition 4.14. Lemma 4.15 ensures that

$$\begin{aligned} & \|\nabla_{x,z}v\|_{L^\infty(I; H^{k-\frac{1}{2}})} + \|\nabla_{x,z}v\|_{L^2(I; H^k)} \leq K_\eta(\|\nabla_{x,z}v\|_{L^2(\mathcal{S})} + \|(f, f_b)\|_{H^s} \\ & \quad + \|F_0\|_{L_z^2(I; H^{k-1})} + \|\eta\|_{H^{s+\frac{1}{2}}}\|\nabla_{x,z}v\|_{L^\infty(\mathcal{S})}). \end{aligned} \quad (4.20)$$

Let us assume that for $0 \leq \ell \leq k-1$, there holds

$$\begin{aligned} \|\nabla_{x,z}^{\ell+1}v\|_{L^2(I; H^{k-\ell})} & \leq K_\eta(\|\nabla_{x,z}v\|_{L^2(\mathcal{S})} + \|(f, f_b)\|_{H^s} \\ & \quad + \sum_{\ell' \leq \ell-1} \|\nabla_{x,z}^{\ell'} F_0\|_{L^2(\mathcal{S})} + \|\eta\|_{H^{s+\frac{1}{2}}}\|\nabla_{x,z}v\|_{L^\infty(\mathcal{S})}). \end{aligned} \quad (4.21)$$

Next we show that the inequality holds for $\ell = k$. Using the equation (4.11), we get

$$\|\partial_z^{k+1}v\|_{L^2(\mathcal{S})} \leq \|\partial_z^{k-1}(\alpha\Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v)\|_{L^2(\mathcal{S})} + \|\partial_z^{k-1}F_0\|_{L^2(\mathcal{S})}. \quad (4.22)$$

As in the proof of Proposition 4.10, let $(\bar{v}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be an extension of $(v, \alpha, \beta, \gamma)$ to \mathbf{R}^{d+1} keeping the corresponding Sobolev norm. We infer from Lemma 2.11 that

$$\begin{aligned} \|(\bar{\alpha} - 1)\Delta \bar{v}\|_{H^{k-1}} & \leq C\|\bar{\alpha} - 1\|_{L^\infty}\|\nabla^2 \bar{v}\|_{H^{k-1}} + C\|\nabla \bar{v}\|_{L^\infty}\|\bar{\alpha} - 1\|_{H^k} \\ & \leq C\|\alpha - 1\|_{L^\infty(\mathcal{S})}\|\nabla^2 v\|_{H^{k-1}(\mathcal{S})} + C\|\nabla v\|_{L^\infty(\mathcal{S})}\|\alpha - 1\|_{H^k(\mathcal{S})} \end{aligned}$$

Similarly, we have

$$\|\bar{\beta} \nabla \partial_z \bar{v}\|_{H^{k-1}} \leq C\|\beta\|_{L^\infty(\mathcal{S})}\|\nabla \nabla_{x,z}v\|_{H^{k-1}(\mathcal{S})} + C\|\nabla_{x,z}v\|_{L^\infty(\mathcal{S})}\|\beta\|_{H^k(\mathcal{S})}.$$

Using Bony's decomposition (2.3), we write

$$\bar{\gamma} \partial_z \bar{v} = T_{\bar{\gamma}} \partial_z \bar{v} + T_{\partial_z \bar{v}} \bar{\gamma} + R(\partial_z \bar{v}, \bar{\gamma}).$$

It follows from Lemma 2.2 that

$$\begin{aligned} \|S_{j-3}\bar{\gamma}\Delta_j(\partial_z\bar{v})\|_{L^2} &\leq C\|\bar{\gamma}\|_{L_z^\infty(\mathbf{R};C^{-\frac{1}{2}})}\|\langle D\rangle^{\frac{1}{2}}\Delta_j(\partial_z\bar{v})\|_{L^2} \\ &\leq Cc_j2^{-j(k-1)}\|\bar{\gamma}\|_{L_z^\infty(\mathbf{R};C^{-\frac{1}{2}})}\|\langle D\rangle^{\frac{1}{2}}\partial_z\bar{v}\|_{H^{k-1}} \end{aligned}$$

with $\|\{c_j\}\|_{\ell^2} \leq 1$. Then Lemma 2.4 ensures that

$$\|T_{\bar{\gamma}}\partial_z\bar{v}\|_{H^{k-1}} \leq C\|\bar{\gamma}\|_{L_z^\infty(\mathbf{R};C^{-\frac{1}{2}})}\|\langle D\rangle^{\frac{1}{2}}\partial_z\bar{v}\|_{H^{k-1}}.$$

By Lemma 2.10, we have

$$\|T_{\partial_z\bar{v}}\bar{\gamma}\|_{H^{k-1}} \leq C\|\partial_zv\|_{L^\infty(\bar{\mathcal{S}})}\|\gamma\|_{H^{k-1}(\bar{\mathcal{S}})},$$

and for $k > 1$, we have

$$\|R(\partial_z\bar{v}, \bar{\gamma})\|_{H^{k-1}} \leq C\|\partial_zv\|_{L^\infty(\bar{\mathcal{S}})}\|\gamma\|_{H^{k-1}(\bar{\mathcal{S}})}.$$

This proves that for $k > 1$,

$$\|\bar{\gamma}\partial_z\bar{v}\|_{H^{k-1}} \leq C\|\partial_zv\|_{L^\infty(\bar{\mathcal{S}})}\|\gamma\|_{H^{k-1}(\bar{\mathcal{S}})} + C\|\gamma\|_{L_z^\infty(I;C^{-\frac{1}{2}})}\|\langle D\rangle^{\frac{1}{2}}\partial_zv\|_{H^{k-1}(\bar{\mathcal{S}})}.$$

For $k = 1$, it is obvious that

$$\|\bar{\gamma}\partial_z\bar{v}\|_{L^2} \leq C\|\gamma\|_{L_z^2(I;L^\infty)}\|\partial_zv\|_{L_z^\infty(I;L^2)}.$$

The above estimates together with Lemma 4.4 and Lemma 4.5 ensure that

$$\begin{aligned} &\|\partial_z^{k-1}(\alpha\Delta v + \beta \cdot \nabla \partial_zv - \gamma\partial_zv)\|_{L^2(\bar{\mathcal{S}})} \\ &\leq C\|\nabla_{x,z}v\|_{L^\infty(\bar{\mathcal{S}})}(\|(\alpha-1, \beta)\|_{H^k(\bar{\mathcal{S}})} + \|\gamma\|_{H^{k-1}(\bar{\mathcal{S}})}) + C\|(\alpha, \beta)\|_{L^\infty(\bar{\mathcal{S}})}\|\nabla\nabla_{x,z}v\|_{H^{k-1}(\bar{\mathcal{S}})} \\ &\quad + C\|\gamma\|_{L_z^\infty(I;C^{-\frac{1}{2}})}\|\langle D\rangle^{\frac{1}{2}}\partial_zv\|_{H^{k-1}(\bar{\mathcal{S}})} + C\|\gamma\|_{L_z^2(I;L^\infty)}^2\|\partial_zv\|_{L_z^\infty(I;L^2)} \\ &\leq C\|\nabla_{x,z}v\|_{L^\infty(\bar{\mathcal{S}})}\|\eta\|_{H^s} + K_\eta(\|\nabla\nabla_{x,z}v\|_{H^{k-1}(\bar{\mathcal{S}})} + \|\langle D\rangle^{\frac{1}{2}}\partial_zv\|_{H^{k-1}(\bar{\mathcal{S}})} + \|\partial_zv\|_{L_z^\infty(I;L^2)}), \end{aligned}$$

from which and (4.22), we infer that

$$\begin{aligned} \|\partial_z^{k+1}v\|_{L^2(\bar{\mathcal{S}})} &\leq C\|\nabla_{x,z}v\|_{L^\infty(\bar{\mathcal{S}})}\|\eta\|_{H^s} \\ &\quad + K_\eta(\|\nabla\nabla_{x,z}v\|_{H^{k-1}(\bar{\mathcal{S}})} + \|\langle D\rangle^{\frac{1}{2}}\partial_zv\|_{H^{k-1}(\bar{\mathcal{S}})} + \|\partial_zv\|_{L_z^\infty(I;L^2)}) + \|\partial_z^{k-1}F_0\|_{L^2(\bar{\mathcal{S}})}. \end{aligned}$$

which along with (4.20) and (4.21) implies the proposition by the interpolation. \square

4.5. Elliptic estimates in Besov space. In this subsection, we present the elliptic estimates in Besov space which will be used in the proof of break-down criterion.

Proposition 4.16. *Let $I_0 = [a, 0]$ and $I_1 = [b, 0]$ with $b < a$. Let $q \in [1, \infty]$, $r \in [0, \frac{1}{2}]$. Assume that $v \in X_q^r(I_1)$ is a solution of the elliptic equation*

$$\partial_z^2v + \alpha\Delta v + \beta \cdot \nabla \partial_zv - \gamma\partial_zv = F_{00} + \partial_zG_0 \quad \text{in } \bar{\mathcal{S}}$$

with $v(x, 0) = f(x)$. Then it holds that for any $\delta > 0$,

$$\begin{aligned} \|\nabla_{x,z}v\|_{X_q^r(I_0)} &\leq K_\eta \left(\|f\|_{B_{\infty,q}^{r+1}} + \|F_{00}\|_{Y_q^r(I)} + \|G_0\|_{X_q^r(I)} \right. \\ &\quad \left. + \|\nabla_{x,z}v\|_{L_z^\infty(I_0;C^{-\delta})} + \|\nabla_{x,z}v\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\delta}{2}})} \right). \end{aligned}$$

Here $X_q^r(I_0) = \tilde{L}_z^\infty(I_0; B_{\infty,q}^r)$.

Let us recall the following lemma from [32], which can be proved by using Bony's decomposition (2.3), Lemma 2.17 and Lemma 2.18.

Lemma 4.17. *Let $I_0 = [a, 0]$ and $I_1 = [b, 0]$ with $b < a$. Then for any $r \leq \frac{1}{2}$ and $q \in [1, \infty]$, it holds that*

$$\begin{aligned} \|F_1\|_{\tilde{L}_z^2(I_1; B_{\infty, q}^{r-\frac{1}{2}})} &\leq K_\eta (\|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; C^{-\frac{\epsilon}{2}})} + \|\partial_z v\|_{\tilde{L}_z^\infty(I_0; B_{\infty, q}^{r-\frac{1}{2}})} \\ &\quad + \|\nabla_{x,z} v\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}), \\ \|F_2\|_{\tilde{L}_z^2(I_1; B_{\infty, q}^{r-\frac{1}{2}})} &\leq K_\eta (\|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; C^{-\frac{\epsilon}{2}})} + \|\nabla_{x,z} v\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}), \\ \|F_3\|_{\tilde{L}_z^2(I_1; B_{\infty, q}^{r-\frac{1}{2}})} &\leq K_\eta (\|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; C^{-\frac{\epsilon}{2}})} + \|\nabla_{x,z} v\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}). \end{aligned}$$

Proof of Proposition 4.16. Let χ be a smooth function supported in I_1 and $\chi(z) = 1$ for $z \in I_0$. Set $w = \chi(z)(\partial_z - T_A)v$. Then (w, v) satisfies

$$\begin{aligned} \partial_z w - T_A w &= \tilde{F}, \quad w(b) = 0, \\ \partial_z v - T_A v &= w \quad \text{in } [a, 0], \quad v(0) = f, \end{aligned}$$

where $\tilde{F} = \chi(z)(F_{00} + \partial_z G_0 + F_1 + F_2 + F_3) + \chi'(z)(\partial_z - T_A)v$.

Let $w_1 = w - \chi(z)G_0$. Then w_1 satisfies

$$\partial_z w_1 - T_A w_1 = \tilde{F} - \partial_z(\chi(z)G_0) - \chi(z)T_A G_0, \quad w(b) = 0.$$

Then Proposition 3.2 together with Lemma 4.17 and Proposition 2.7 ensures that

$$\begin{aligned} \|w - G_0\|_{X_{2,q}^r(I_0)} &\leq K_\eta (\|\tilde{F} - \chi(z)\partial_z G_0\|_{Y_q^r(I_1)} + \|G_0\|_{X_q^r(I_1)} + \|w\|_{\tilde{L}_z^2(I_1; C^{-\delta})}) \\ &\leq K_\eta (\|F_{00}\|_{Y_q^r(I)} + \|G_0\|_{X_q^r(I_1)} + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; B_{\infty, q}^{r-\frac{1}{2}})} \\ &\quad + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; C^{-\frac{\epsilon}{2}})} + \|\nabla_{x,z} v\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}). \end{aligned}$$

Note that $(\partial_z - T_A)\nabla v = \nabla(w - G_0) + \nabla G_0 + T_{\nabla A}v$ on I_0 . It follows from Proposition 3.2, Lemma 4.6 and Remark 2.8 that

$$\begin{aligned} \|\nabla v\|_{X_q^r(I_0)} &\leq K_\eta (\|f\|_{B_{\infty, q}^{r+1}} + \|w - G_0\|_{\tilde{L}_z^2(I_0; B_{\infty, q}^{r+\frac{1}{2}})} + \|G_0\|_{X_q^r(I_1)} + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; B_{\infty, q}^{r-\frac{1}{2}})} + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; C^{-\delta})}) \\ &\leq K_\eta (\|f\|_{B_{\infty, q}^{r+1}} + \|F_{00}\|_{Y_q^r(I)} + \|G_0\|_{X_q^r(I_1)} + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; B_{\infty, q}^{r-\frac{1}{2}})} \\ &\quad + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; C^{-\frac{\epsilon}{2}})} + \|\nabla_{x,z} v\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}). \end{aligned}$$

The same estimate for $\partial_z v$ can be deduced by using $\partial_z v = T_A v + w$ and Proposition 2.7. Thus, we obtain

$$\begin{aligned} \|\nabla_{x,z} v\|_{X_q^r(I_0)} &\leq K_\eta (\|f\|_{B_{\infty, q}^{r+1}} + \|F_{00}\|_{Y_q^r(I)} + \|G_0\|_{X_q^r(I_1)} + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; B_{\infty, q}^{r-\frac{1}{2}})} \\ &\quad + \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; C^{-\frac{\epsilon}{2}})} + \|\nabla_{x,z} v\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}). \end{aligned}$$

This together with the interpolation inequality

$$\|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; B_{\infty, q}^{r-\frac{1}{2}} \cap C^{-\frac{\epsilon}{2}})} \leq K_\eta \|\nabla_{x,z} v\|_{L_z^\infty(I_0; C^{-\delta})} + \frac{1}{4K_\eta} \|\nabla_{x,z} v\|_{\tilde{L}_z^\infty(I_0; B_{\infty, q}^r)}$$

implies the proposition. \square

4.6. Interior $W^{1,p}$ estimate. We consider the elliptic equation

$$\Delta_{x,y}\phi = \operatorname{div}_{x,y}g \quad \text{in } \mathcal{S}. \quad (4.23)$$

Given a point $X_0 = (x_0, y_0) \in \mathbf{R}^d \times \mathbf{R}$, let $B_r(X_0)$ be a ball of radius r centered at X_0 . We have the following interior $W^{1,p}$ estimate.

Proposition 4.18. *Suppose that $\phi \in H^1(\mathcal{S})$ is a solution of (4.23) with $g \in L^p(\mathcal{S})$ for $p \geq 2$. Let $I_x = [-1 + c_1 h_0, \eta(x) - c_2 h_0]$, where $c_1, c_2 > 0$ and $c_1 + c_2 < 1$. Then there exists $\delta_1 > 0$ depending only on $\|\eta\|_{C^{1+\varepsilon}}, h_0$ and c_1, c_2 so that $B_{\delta_1}(X_0) \subseteq \mathcal{S}$ for any $X_0 \in \{(x, y) : x \in \mathbf{R}^d, y \in I_x\}$ and*

$$\|\phi\|_{W^{1,p}(B_{\delta_1/2}(X_0))} \leq K_\eta (\|g\|_{L^p(B_{\delta_1}(X_0))} + \|\nabla_{x,y}\phi\|_{L^2(B_{\delta_1}(X_0))}).$$

Proof. Given any point $X_0 \in \mathbf{R}^d \times I_x$. Let $\bar{\phi}$ be a solution of the elliptic equation in $B_{\delta_1}(X_0)$:

$$\Delta_{x,y}\bar{\phi} = \operatorname{div}_{x,y}g \quad \text{in } B_{\delta_1}(X_0), \quad \bar{\phi}|_{\partial B_{\delta_1}(X_0)} = 0.$$

The classical $W^{1,p}$ elliptic estimate ensures that

$$\|\bar{\phi}\|_{W^{1,p}(B_{\delta_1}(X_0))} \leq C(\delta_1) \|g\|_{L^p(B_{\delta_1}(X_0))}.$$

Then by using the interior gradient estimate of the harmonic function $\phi - \bar{\phi}$, we deduce that

$$\begin{aligned} \|\phi\|_{W^{1,p}(B_{\delta_1/2}(X_0))} &\leq \|\phi - \bar{\phi}\|_{W^{1,p}(B_{\delta_1/2}(X_0))} + \|\bar{\phi}\|_{W^{1,p}(B_{\delta_1/2}(X_0))} \\ &\leq C(\delta_1) (\|\nabla_{x,y}(\phi - \bar{\phi})\|_{L^2(B_{\delta_1}(X_0))} + \|\bar{\phi}\|_{W^{1,p}(B_{\delta_1}(X_0))}) \\ &\leq C(\delta_1) (\|\nabla_{x,y}\phi\|_{L^2(B_{\delta_1}(X_0))} + \|g\|_{L^p(B_{\delta_1}(X_0))}). \end{aligned}$$

The proof is finished. \square

5. DIRICHLET-NEUMANN OPERATOR

In this section, we assume that $\eta \in C^{\frac{3}{2}+\varepsilon}(\mathbf{R}^d)$ for some $\varepsilon > 0$ and satisfies (4.1). We will use some notations from section 4.

5.1. Definition and parilinearization. We consider the boundary value problem

$$\begin{cases} \Delta_{x,y}\phi = 0 & \text{in } \mathcal{S}, \\ \phi|_{y=\eta(x)} = f, \quad \phi|_{y=-1} = 0. \end{cases} \quad (5.1)$$

where $\mathcal{S} = \{(x, y) : x \in \mathbf{R}^d, -1 < y < \eta(x)\}$. Given $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$, the existence of the variation solution ϕ with $\nabla_{x,y}\phi \in L^2(\mathcal{S})$ can be deduced by using Riesz theorem, see [3] for example. Moreover, it holds that

$$\|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})} \leq C(\|\eta\|_{W^{1,\infty}}, h_0) \|f\|_{H^{\frac{1}{2}}}. \quad (5.2)$$

Let $\eta \in H^s(\mathbf{R}^d)$ for $s > \frac{d}{2} + \frac{3}{2}$. This together with Proposition 4.10 yields that for any $\sigma \in [-\frac{1}{2}, s-1]$,

$$\|\tilde{\phi}\|_{H^{\sigma+\frac{3}{2}}(\overline{\mathcal{S}})} \leq P_{s,\eta} \|f\|_{H^{\sigma+1}}. \quad (5.3)$$

Definition 5.1. *Given η, f, ϕ as above, the Dirichlet-Neumann(DN) operator $G(\eta)$ is defined by*

$$G(\eta)f \stackrel{\text{def}}{=} \sqrt{1 + |\nabla\eta|^2} \partial_n \phi|_{y=\eta}.$$

The DN operator is a positive self-adjoint operator. More precisely, for any $f, g \in H^{\frac{1}{2}}(\mathbf{R}^d)$, we have

$$\begin{aligned}\langle G(\eta)f, g \rangle &= \langle f, G(\eta)g \rangle, \\ \langle G(\eta)f, f \rangle &= \|\nabla_{x,y}\phi\|_{L^2(\mathcal{S})} \geq c\|f\|_{H^{\frac{1}{2}}}^2.\end{aligned}$$

In terms of $\tilde{\phi}$, the Dirichlet-Neumann operator $G(\eta)$ can be written as

$$G(\eta)f = \left(\frac{1 + |\nabla\rho_\delta|^2}{\partial_z\rho_\delta} \partial_z\tilde{\phi} - \nabla\rho_\delta \cdot \nabla\tilde{\phi} \right) \Big|_{z=0}.$$

Following [3], we first parilinearize $G(\eta)$. We set

$$\zeta_1 = \frac{1 + |\nabla\rho_\delta|^2}{\partial_z\rho_\delta} \Big|_{z=0}, \quad \zeta_2 = \nabla\rho_\delta \Big|_{z=0}.$$

It is easy to show that

$$\|\zeta_1 - 1\|_{H^{s-1}} + \|\zeta_2\|_{H^{s-1}} \leq K_\eta \|\eta\|_{H^s}. \quad (5.4)$$

Using Bony's decomposition (2.3), we decompose $G(\eta)$ as

$$\begin{aligned}G(\eta)f &= \partial_z\tilde{\phi} + T_{\zeta_1-1}\partial_z\tilde{\phi} + T_{\partial_z\tilde{\phi}}(\zeta_1 - 1) + R(\zeta_1 - 1, \partial_z\tilde{\phi}) - T_{i\zeta_2 \cdot \xi}\tilde{\phi} \\ &\quad - T_{\nabla\tilde{\phi}} \cdot \zeta_2 - R(\zeta_2, \nabla\tilde{\phi}) \Big|_{z=0}.\end{aligned}$$

Replacing $\partial_z\tilde{\phi}$ by $T_A\tilde{\phi}$, we get

$$G(\eta)f = T_\lambda f + R(\eta)f, \quad (5.5)$$

where $\lambda = \zeta_1 A - i\zeta_2 \cdot \xi \Big|_{z=0}$ with

$$A = \frac{1}{2}(-i\beta \cdot \xi + \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}), \quad (5.6)$$

and $R(\eta)$ is the remainder of DN operator given by

$$\begin{aligned}R(\eta)f &= \left[(T_{\zeta_1}T_A - T_{\zeta_1 A})\tilde{\phi} - T_{\zeta_1}(\partial_z - T_A)\tilde{\phi} \right. \\ &\quad \left. + (S_2(\partial_z\tilde{\phi}) + T_{\partial_z\tilde{\phi}}(\zeta_1 - 1) + R(\zeta_1 - 1, \partial_z\tilde{\phi}) - T_{\nabla\tilde{\phi}} \cdot \zeta_2 - R(\nabla\tilde{\phi}, \zeta_2)) \right] \Big|_{z=0} \\ &\triangleq R_1(\eta)f + R_2(\eta)f + R_3(\eta)f. \quad (5.7)\end{aligned}$$

5.2. Sobolev estimate of the remainder. In this subsection, we present Sobolev estimates for the remainder in the case when the boundary is smooth, which will be used in the proof of the uniform estimates of the approximate solutions.

Proposition 5.2. *Assume that $\eta \in H^s(\mathbf{R}^d)$ for $s > \frac{d}{2} + 2$. Then it holds that*

$$\|R(\eta)f\|_{H^{s-1}} \leq P_{s,\eta} \|f\|_{H^{s-1}}.$$

Proof. Thanks to the fact that for $s > \frac{d}{2} + 2$,

$$M_1^0(\zeta_1) + \mathcal{M}_1^1(A) \leq P_{s,\eta},$$

we deduce from Proposition 2.6 and (5.3) that

$$\|R_1(\eta)f\|_{H^{s-1}} \leq P_{s,\eta} \|\nabla\tilde{\phi}\|_{L_z^\infty(I; H^{s-2})} \leq P_{s,\eta} \|f\|_{H^{s-1}}.$$

By Lemma 2.10 and (5.4), we have

$$\|R_3(\eta)f\|_{H^{s-1}} \leq C \|\nabla_{x,z}\tilde{\phi}\|_{L_z^\infty(I; L^2)} + P_{s,\eta} \|\nabla_{x,z}\tilde{\phi}\|_{L^\infty(\overline{\mathcal{S}})}$$

$$\leq P_{s,\eta} \|\nabla_{x,z} \tilde{\phi}\|_{L_z^\infty(I; H^{s-2})} \leq P_{s,\eta} \|f\|_{H^{s-1}}.$$

For $R_2(\eta)$, we get by Lemma 2.10 that

$$\|R_2(\eta)f\|_{H^{s-1}} \leq P_{s,\eta} \|(\partial_z - T_A)\tilde{\phi}\|_{L_z^\infty([-\frac{1}{2}, 0]; H^{s-1})}. \quad (5.8)$$

While by the proof of Proposition 4.12, we know that $w = \chi(z)(\partial_z - T_A)\tilde{\phi}$ satisfies

$$\partial_z w - T_a w = F', \quad w(-1) = 0.$$

where $F' = \chi(z)(F_1 + F_2 + F_3) + \chi'(z)(\partial_z - T_A)\tilde{\phi}$ and χ is a smooth function satisfying $\chi(-1) = 0$ and $\chi(z) = 1$ for $z \in [-\frac{1}{2}, 0]$. Then it follows from Proposition 3.1, Lemma 5.3 and (5.3) that

$$\begin{aligned} \|(\partial_z - T_A)\tilde{\phi}\|_{X^{s-1}([-\frac{1}{2}, 0])} &\leq \|w\|_{X^{s-1}(I)} \\ &\leq P_{s,\eta} \left(\sum_{i=1}^3 \|F_i\|_{L_z^2(I; H^{s-\frac{3}{2}})} + \|\nabla_{x,z} \tilde{\phi}\|_{X^{s-2}(I)} \right) \\ &\leq P_{s,\eta} \|f\|_{H^{s-1}}, \end{aligned}$$

which along with (5.8) gives

$$\|R_2(\eta)f\|_{H^{s-1}} \leq P_{s,\eta} \|f\|_{H^{s-1}}.$$

Putting the estimates of $R_i(\eta)f$ together concludes the proposition. \square

Lemma 5.3. *Let $s > \frac{d}{2} + 2$. It holds that for $i = 1, 2, 3$,*

$$\|F_i\|_{L_z^2(I; H^{s-\frac{3}{2}})} \leq P_{s,\eta} \|\nabla_{x,z} \tilde{\phi}\|_{X^{s-2}(I)}.$$

Proof. Recall that $F_1 = \gamma \partial_z \tilde{v}$. It follows from Lemma 2.13 that

$$\begin{aligned} \|F_1\|_{L_z^2(I; H^{s-\frac{3}{2}})} &\leq C \|\gamma\|_{L^\infty} \|\partial_z \tilde{\phi}\|_{L_z^2(I; H^{s-\frac{3}{2}})} + C \|\partial_z \tilde{\phi}\|_{L^\infty} \|\gamma\|_{L_z^2(I; H^{s-\frac{3}{2}})} \\ &\leq C \|\gamma\|_{L_z^\infty(I; H^{s-2})} \|\partial_z \tilde{\phi}\|_{L_z^2(I; H^{s-\frac{3}{2}})} + C \|\partial_z \tilde{\phi}\|_{L_z^\infty(I; H^{s-2})} \|\gamma\|_{L_z^2(I; H^{s-\frac{3}{2}})} \\ &\leq P_{s,\eta} \|\nabla_{x,z} \tilde{\phi}\|_{X^{s-2}(I)}. \end{aligned}$$

Recall that

$$F_2 = (T_\alpha - \alpha)\Delta \tilde{\phi} + (T_\beta - \beta) \cdot \nabla \partial_z \tilde{\phi}.$$

Then we get by Lemma 2.10 that

$$\begin{aligned} \|F_2\|_{L_z^2(I; H^{s-\frac{3}{2}})} &\leq C \|\nabla_{x,z} \tilde{\phi}\|_{L^\infty} (1 + \|\alpha - 1\|_{L_z^2(I; H^{s-\frac{1}{2}})} + \|\beta\|_{L_z^2(I; H^{s-\frac{1}{2}})}) \\ &\leq P_{s,\eta} \|\nabla_{x,z} \tilde{\phi}\|_{X^{s-2}(I)}. \end{aligned}$$

Recall that

$$F_3 = (T_a T_A - T_\alpha \Delta) \tilde{\phi} - (T_a + T_A + T_\beta \cdot \nabla) \partial_z \tilde{\phi} - T_{\partial_z A} \tilde{\phi}.$$

For $s > \frac{d}{2} + 2$, we have

$$\mathcal{M}_1^0(a) + \mathcal{M}_1^1(A) \leq P_{s,\eta}.$$

Then we deduce from Lemma 4.6 and Proposition 2.6 that

$$\|F_3\|_{L_z^2(I; H^{s-\frac{3}{2}})} \leq P_{s,\eta} \|\nabla_{x,z} \tilde{\phi}\|_{X^{s-2}(I)}.$$

The proof is finished. \square

5.3. Tame estimate of the remainder. In this subsection, we present tame estimates for the remainder in the case when the boundary has more one half derivative than f . The result will be used in the proof of the break-down criterion.

Proposition 5.4. *Assume that $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ for $s > \frac{d}{2} + 1$. Let I_1 be as in Proposition 4.12 and $\tilde{S} = \mathbf{R}^d \times I_1$. It holds that*

$$\begin{aligned} \|R(\eta)f\|_{H^{s-\frac{1}{2}}} &\leq K_\eta(\|f\|_{H^s} + \|\nabla_{x,z}\tilde{\phi}\|_{L^\infty(\tilde{S})}\|\eta\|_{H^{s+\frac{1}{2}}}), \\ \|R(\eta)f\|_{H^{s-1}} &\leq K_\eta(\|f\|_{H^{s-\frac{1}{2}}} + \|\nabla_{x,z}\tilde{\phi}\|_{L_z^\infty(I_1;C^0)}\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

Proof. Note that $A \in \Gamma_{\frac{1}{2}+\varepsilon}^1(I \times \mathbf{R}^d)$ and $\zeta_1 \in \Gamma_{\frac{1}{2}+\varepsilon}^0(\mathbf{R}^d)$ with the bound

$$\mathcal{M}_{\frac{1}{2}+\varepsilon}^1(A) + M_{\frac{1}{2}+\varepsilon}^0(\zeta_1) \leq K_\eta. \quad (5.9)$$

Then we get by Proposition 2.6, (5.4), Proposition 4.12 and (5.2) that

$$\begin{aligned} \|R_1(\eta)f\|_{H^{s-\frac{1}{2}}} &\leq K_\eta\|\nabla\tilde{\phi}\|_{L_z^\infty([z_0,0];H^{s-1})} \\ &\leq K_\eta(\|\nabla_{x,z}\tilde{\phi}\|_{L^2(\tilde{S})} + \|f\|_{H^s} + \|\nabla_{x,z}\tilde{\phi}\|_{L_z^\infty(I_1;C^0)}\|\eta\|_{H^{s+\frac{1}{2}}}) \\ &\leq K_\eta(\|f\|_{H^s} + \|\nabla_{x,z}\tilde{\phi}\|_{L_z^\infty(I_1;C^0)}\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

Similarly, we deduce from Lemma 2.10 that

$$\begin{aligned} \|R_3(\eta)f\|_{H^{s-\frac{1}{2}}} &\leq K_\eta\|\nabla_{x,z}\tilde{\phi}\|_{L^\infty(\tilde{S})}\|\eta\|_{H^{s+\frac{1}{2}}} + \|\partial_z\tilde{\phi}\|_{L_z^\infty([z_0,0];H^{s-1})} \\ &\leq K_\eta(\|f\|_{H^s} + \|\nabla_{x,z}\tilde{\phi}\|_{L^\infty(\tilde{S})}\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

While by the proof of (4.19), we see that

$$\begin{aligned} \|(\partial_z - T_A)\tilde{\phi}\|_{L_z^\infty([z_0,0];H^{s-\frac{1}{2}})} &\leq K_\eta(\|\nabla_{x,z}\tilde{\phi}\|_{X^{s-1}([z_0,0])} + \|\nabla_{x,z}\tilde{\phi}\|_{L^\infty(\tilde{S})}\|\eta\|_{H^{s+\frac{1}{2}}}) \\ &\leq K_\eta(\|f\|_{H^s} + \|\nabla_{x,z}\tilde{\phi}\|_{L^\infty(\tilde{S})}\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

Then we get by Lemma 2.10 that

$$\begin{aligned} \|R_2(\eta)f\|_{H^{s-\frac{1}{2}}} &\leq K_\eta\|(\partial_z - T_A)\tilde{\phi}\|_{L_z^\infty([z_0,0];H^{s-\frac{1}{2}})} \\ &\leq K_\eta(\|f\|_{H^s} + \|\nabla_{x,z}\tilde{\phi}\|_{L^\infty(\tilde{S})}\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

This completes the proof of the first inequality. The second inequality can be proved similarly. \square

5.4. Hölder estimate of the remainder.

Proposition 5.5. *Let $I_0 = [-\frac{1}{2}, 0]$ and $I_1 = [-\frac{3}{4}, 0]$. It holds that for any $\delta > 0$,*

$$\|R(\eta)f\|_{C^{\frac{1}{2}}} \leq K_\eta(\|f\|_{C^1} + \|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^\infty(I_0;C^{-\delta})} + \|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\delta}{2}})}).$$

Proof. Thanks to (5.9), we get by Proposition 2.7 that

$$\|R_1(\eta)f\|_{C^{\frac{1}{2}}} \leq K_\eta\|\nabla\tilde{\phi}\|_{L_z^\infty(I_0;C^0)}.$$

We can see from the proof of Proposition 4.16 that

$$\|(\partial_z - T_A)\tilde{\phi}\|_{\tilde{L}_z^\infty([-\frac{1}{4},0];C^{\frac{1}{2}})} \leq K_\eta(\|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^\infty(I_0;C^0)} + \|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\delta}{2}})}),$$

which together with Lemma 2.10 gives

$$\|R_2(\eta)f\|_{C^{\frac{1}{2}}} \leq K_\eta \|(\partial_z - T_A)\tilde{\phi}\|_{L_z^\infty([- \frac{1}{4}, 0]; C^{\frac{1}{2}})} \leq K_\eta (\|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^\infty(I_0; C^0)} + \|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}).$$

By Lemma 2.10 again, we get

$$\|R_3(\eta)f\|_{C^{\frac{1}{2}}} \leq K_\eta \|\nabla_{x,z}\tilde{\phi}\|_{L_z^\infty(I; C^0)}.$$

This together with Proposition 4.16 shows that

$$\begin{aligned} \|R(\eta)f\|_{C^{\frac{1}{2}}} &\leq K_\eta (\|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^\infty(I_0; C^0)} + \|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}) \\ &\leq K_\eta (\|f\|_{B_{\infty,\infty}^1} + \|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^\infty(I_0; C^{-\delta})} + \|\nabla_{x,z}\tilde{\phi}\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}). \end{aligned}$$

This finishes the proof. \square

6. NEW FORMULATION AND PARALINEARIZATION

6.1. New formulation. First of all, we derive the evolution equations for the free surface and the trace of the velocity on the boundary. We denote

$$\begin{aligned} V &\triangleq (v^1, \dots, v^d)|_{y=\eta}, \quad V_b \triangleq (v^1, \dots, v^d)|_{y=-1}, \quad B \triangleq v^{d+1}|_{y=\eta}, \\ a &\triangleq -\partial_y P|_{y=\eta}, \quad \zeta \triangleq \nabla \eta. \end{aligned}$$

Using the fact that for any function $f = f(t, x, y)$,

$$(\partial_t + V \cdot \nabla)(f|_{y=\eta}) = (\partial_t f + v \cdot \nabla_{x,y} f)|_{y=\eta},$$

we deduce from (1.1) that

$$(\partial_t + V \cdot \nabla)B = a - 1, \tag{6.1}$$

$$(\partial_t + V \cdot \nabla)V + a\zeta = 0, \tag{6.2}$$

$$(\partial_t + V_b \cdot \nabla)V_b + \nabla P|_{y=-1} = 0. \tag{6.3}$$

Let ω be the vorticity of the fluid, which is defined by

$$\omega = (\omega_{i,j})_{1 \leq i,j \leq d+1}, \quad \omega_{i,j} = \partial_{x_i} v^j - \partial_{x_j} v^i.$$

The motion of the fluid is determined by the vorticity equation

$$\omega_t + v \cdot \nabla_{x,y} \omega = \omega \otimes \nabla_{x,y} v \quad \text{in } \Omega_t. \tag{6.4}$$

Here $(\omega \otimes \nabla_{x,y} v)_{i,j} = \omega_{k,i} \partial_j v^k + \omega_{k,j} \partial_i v^k$.

The velocity v can be recovered from the vorticity ω by solving the elliptic equation

$$\begin{cases} \Delta_{x,y} v = -\nabla_{x,y} \times \omega & \text{in } \Omega_t, \\ v|_{y=\eta} = (V, B), \quad v|_{y=-1} = (V_b, 0). \end{cases} \tag{6.5}$$

where $\omega = (\omega_{i,j})_{1 \leq i,j \leq d+1}$ with $\omega_{i,j} = \partial_{x_i} v^j - \partial_{x_j} v^i$. The pressure P of the fluid is determined by solving the following elliptic equation:

$$\begin{cases} -\Delta_{x,y} P = \partial_i v^j \partial_j v^i & \text{in } \Omega_t, \\ P|_{y=\eta} = 0, \quad \partial_y P|_{y=-1} = -1. \end{cases} \tag{6.6}$$

We decompose the velocity v into the irrotational part v_{ir} and the rotational part v_ω , i.e.,

$$\begin{cases} \Delta_{x,y} v_{ir} = 0 & \text{in } \Omega_t, \\ v_{ir}|_{y=\eta} = (V, B), \quad v_{ir}|_{y=-1} = 0, \end{cases} \tag{6.7}$$

and

$$\begin{cases} \Delta_{x,y} v_\omega = -\nabla_{x,y} \times \omega & \text{in } \Omega_t, \\ v_\omega|_{y=\eta} = 0, \quad v_\omega|_{y=-1} = (V_b, 0). \end{cases} \quad (6.8)$$

It follows from (1.4) that

$$(\partial_t + V \cdot \nabla) \zeta = \nabla B - \sum_j \nabla V_j \zeta^j. \quad (6.9)$$

Then a direct calculation yields

$$\begin{aligned} & \partial_{x_i} B - \partial_{x_i} V_j \partial_{x_j} \eta \\ &= \partial_{x_i} v^{d+1} + \partial_{x_i} \eta \partial_y v^{d+1} - \partial_{x_j} \eta (\partial_{x_i} v^j + \partial_{x_i} \eta \partial_y v^j)|_{y=\eta} \\ &= (\partial_y v^i - \partial_{x_j} v^i \cdot \partial_{x_j} \eta) + \partial_{x_i} \eta (\partial_y v^{d+1} - \partial_{x_j} \eta \partial_{x_j} v^{d+1}) \\ &\quad + (\omega_{i,d+1} - \partial_{x_j} \eta \omega_{ij} + \partial_{x_i} \eta \partial_{x_j} \eta \omega_{j,d+1})|_{y=\eta} \\ &= (\partial_y v_{ir}^i - \partial_{x_j} v_{ir}^i \cdot \partial_{x_j} \eta) + \partial_{x_i} \eta (\partial_y v_{ir}^{d+1} - \partial_{x_j} \eta \partial_{x_j} v_{ir}^{d+1}) \\ &\quad + (\partial_y v_\omega^i - \partial_{x_j} v_\omega^i \cdot \partial_{x_j} \eta) + \partial_{x_i} \eta (\partial_y v_\omega^{d+1} - \partial_{x_j} \eta \partial_{x_j} v_\omega^{d+1}) \\ &\quad + (\omega_{i,d+1} - \partial_{x_j} \eta \omega_{ij} + \partial_{x_i} \eta \partial_{x_j} \eta \omega_{j,d+1})|_{y=\eta} \\ &= G(\eta) V_i + \partial_{x_i} \eta G(\eta) B + R_\omega^i, \end{aligned}$$

where

$$\begin{aligned} R_\omega^i &\triangleq (\partial_y v_\omega^i - \partial_{x_j} v_\omega^i \cdot \partial_{x_j} \eta) + \partial_{x_i} \eta (\partial_y v_\omega^{d+1} - \partial_{x_j} \eta \partial_{x_j} v_\omega^{d+1}) \\ &\quad + (\omega_{i,d+1} - \partial_{x_j} \eta \omega_{ij} + \partial_{x_i} \eta \partial_{x_j} \eta \omega_{j,d+1})|_{y=\eta}. \end{aligned}$$

Thus, $\zeta = \nabla \eta$ satisfies

$$(\partial_t + V \cdot \nabla) \zeta = G(\eta) V + \zeta G(\eta) B + R_\omega. \quad (6.10)$$

The term R_ω induced by the vorticity will lead the system to lose one half derivative.

6.2. Paralinearization. We paralinearize the system (6.1), (6.2) and (6.10). For this end, we introduce so called good unknown $U = V + T_\zeta B$. Applying Bony's decomposition and (5.5) to (6.1), (6.2) and (6.10), we obtain

$$\begin{cases} (\partial_t + T_V \cdot \nabla) V + T_a \zeta + T_\zeta (\partial_t + T_V \cdot \nabla) B = h_1, \\ (\partial_t + T_V \cdot \nabla) \zeta = T_\lambda U + h_2 + R_\omega, \end{cases} \quad (6.11)$$

where

$$\begin{aligned} h_1 &\triangleq (T_V - V) \cdot \nabla V - R(a, \zeta) + T_\zeta (T_V - V) \cdot \nabla B, \\ h_2 &\triangleq (T_V - V) \cdot \nabla \zeta + [T_\zeta, T_\lambda] B + (\zeta - T_\zeta) T_\lambda B + R(\eta) V + \zeta R(\eta) B. \end{aligned}$$

Let $D_t \triangleq \partial_t + T_V \cdot \nabla$. In terms of good unknown, the first equation of (6.11) can be rewritten as

$$D_t U + T_a \zeta = h_1 + [D_t, T_\zeta] B. \quad (6.12)$$

Taking D_t on both sides of (6.12), we get

$$D_t^2 U + T_a D_t \zeta = D_t h_1 + [T_a, D_t] \zeta + D_t [D_t, T_\zeta] B,$$

which along with the second equation of (6.11) gives

$$D_t^2 U + T_a \lambda U = f + f_\omega, \quad (6.13)$$

where (f, f_ω) is given by

$$f \triangleq D_t h_1 + (T_{a\lambda} - T_\lambda T_a)U + [T_a, D_t]\zeta + D_t[D_t, T_\zeta]B - T_a h_2, \quad f_\omega \triangleq -T_a R_\omega.$$

Similarly, we have

$$D_t^2 \zeta + T_{a\lambda} \zeta = g + g_\omega, \quad (6.14)$$

where (g, g_ω) is given by

$$g \triangleq D_t h_2 + [D_t, T_\lambda]U + T_\lambda(h_1 + D_t[D_t, T_\zeta]B) + (T_{a\lambda} - T_\lambda T_a)\zeta, \quad g_\omega \triangleq D_t R_\omega.$$

7. ESTIMATE OF THE PRESSURE

The pressure P satisfies

$$\begin{cases} -\Delta_{x,y} P = \partial_i(v^j \partial_j v^i) = \partial_i v^j \partial_j v^i & \text{in } \Omega_t, \\ P|_{y=\eta} = 0, \quad \partial_y P|_{y=-1} = -1. \end{cases} \quad (7.1)$$

Here v is the velocity. In this section, we denote $P_1 \triangleq P + y$.

7.1. H^2 estimate of the pressure.

Lemma 7.1. *Let P be a solution of (7.1). It holds that*

$$\|\nabla_{x,y} P_1\|_{L^2(\Omega_t)} \leq K_\eta (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|v\|_{L^2(\Omega_t)} + \|\eta\|_{H^{\frac{1}{2}}}).$$

Proof. Let $\bar{\eta}(t, x, y) = \chi(y)e^{(y-\eta(t,x))|D|}\eta$ for $y \leq \eta(t, x)$, where χ is a smooth function satisfying $\chi(0) = 1$ and $\chi(y) = 0$ for $y \leq -1 + h_0/2$. Then $P_2 \triangleq P + y - \bar{\eta}$ satisfies

$$\begin{cases} -\Delta_{x,y} P_2 = \Delta_{x,y} \bar{\eta} + \partial_i(v^j \partial_j v^i) & \text{in } \Omega_t, \\ P_2|_{y=\eta} = 0, \quad \partial_y P_2|_{y=-1} = 0. \end{cases}$$

Thanks to $v^{d+1}|_{y=-1} = 0$, we get by integration by parts that

$$\begin{aligned} \int_{\Omega_t} |\nabla_{x,y} P_2|^2 dx dy &= \int_{\Omega_t} (\Delta_{x,y} \bar{\eta} + \partial_i \partial_j (v^j v^i)) P_2 dx dy \\ &= - \int_{\Omega_t} (\partial_j (v^j v^h) + \partial_y (v^h v^{d+1})) \cdot \nabla P_2 dx dy \\ &\quad - \int_{\Omega_t} \partial_y (v^{d+1} v^{d+1}) \partial_y P_1 dx dy - \int_{\Omega_t} \nabla_{x,y} \bar{\eta} \cdot \nabla_{x,y} P_2 dx dy \\ &\leq C (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|v\|_{L^2(\Omega_t)} + \|\nabla_{x,y} \bar{\eta}\|_{L^2(\Omega_t)}) \|\nabla_{x,y} P_2\|_{L^2(\Omega_t)}, \end{aligned}$$

from which, we deduce

$$\|\nabla_{x,y} P_2\|_{L^2(\Omega_t)} \leq C (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|v\|_{L^2(\Omega_t)} + \|\nabla_{x,y} \bar{\eta}\|_{L^2(\Omega_t)}).$$

Then the lemma follows by using the fact that

$$\|\nabla_{x,y} \bar{\eta}\|_{L^2(\Omega_t)} \leq C (\|\eta\|_{W^{1,\infty}}, h_0) \|\eta\|_{H^{\frac{1}{2}}}.$$

This completes the proof. \square

Next, we give the higher estimates of pressure:

Lemma 7.2. *Let P be a solution of (7.1). It holds that*

$$\|\nabla_{x,y} P_1\|_{H^1(\Omega_t)} \leq K_\eta (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|v\|_{H^1(\Omega_t)} + \|\eta\|_{H^{\frac{3}{2}}}).$$

Proof. Let P_2 be as in the proof of Lemma 7.1. Then \tilde{P}_2 satisfies

$$\begin{cases} \partial_z^2 \tilde{P}_2 + \alpha \Delta \tilde{P}_2 + \beta \cdot \nabla \partial_z \tilde{P}_2 - \gamma \partial_z \tilde{P}_2 = F_0, \\ \tilde{P}_2|_{z=0} = 0, \quad \partial_z \tilde{P}_2|_{z=-1} = 0, \end{cases} \quad (7.2)$$

where $F_0 = \alpha \widetilde{\partial_i v^j \partial_j v^i} + \alpha \widetilde{\Delta_{x,y} \eta}$. It follows from (7.2) that

$$\begin{aligned} & \int_{\bar{S}} \partial_z^2 \tilde{P}_2 \Delta \tilde{P}_2 dx dz + \int_{\bar{S}} \alpha |\Delta \tilde{P}_2|^2 dx dz + \int_{\bar{S}} \beta \cdot \nabla \partial_z \tilde{P}_2 \Delta \tilde{P}_2 dx dz \\ &= \int_{\bar{S}} \gamma \partial_z \tilde{P}_2 \Delta \tilde{P}_2 dx dz + \int_{\bar{S}} F_0 \Delta \tilde{P}_2 dx dz. \end{aligned}$$

By integration by parts, we get

$$\int_{\bar{S}} \partial_z^2 \tilde{P}_2 \Delta \tilde{P}_2 dx dz = - \int_{\bar{S}} \partial_z^2 \nabla \tilde{P}_2 \cdot \nabla \tilde{P}_2 dx dz = \int_{\bar{S}} |\nabla \partial_z \tilde{P}_2|^2 dx dz.$$

Thanks to the definition of α, β , it is easy to see that there exists $c > 0$ depending on $\|\eta\|_{W^{1,\infty}}, h_0$ so that

$$\begin{aligned} & \int_{\bar{S}} \partial_z^2 \tilde{P}_2 \Delta \tilde{P}_2 dx dz + \int_{\bar{S}} \alpha |\Delta \tilde{P}_2|^2 dx dz + \int_{\bar{S}} \beta \cdot \nabla \partial_z \tilde{P}_2 \Delta \tilde{P}_2 dx dz \\ & \geq c \int_{\bar{S}} (|\Delta \tilde{P}_2|^2 + |\nabla \partial_z \tilde{P}_2|^2) dx dz. \end{aligned}$$

Hence, we obtain

$$\int_{\bar{S}} (|\Delta \tilde{P}_1|^2 + |\nabla \partial_z \tilde{P}_1|^2) dx dz \leq C (\|\gamma \partial_z \tilde{P}_1\|_{L^2(\bar{S})} + \|F_0\|_{L^2(\bar{S})}) \|\Delta \tilde{P}_1\|_{L^2(\bar{S})}.$$

It follows from Lemma 4.4 that

$$\|\gamma \partial_z \tilde{P}_2\|_{L^2(\bar{S})} \leq C \|\gamma\|_{L_z^2(I; L^\infty)} \|\partial_z \tilde{P}_2\|_{L_z^\infty(I; L^2)} \leq K_\eta \|\partial_z \tilde{P}_2\|_{L^2(\bar{S})}^{\frac{1}{2}} \|\partial_z \nabla \tilde{P}_1\|_{L^2(\bar{S})}^{\frac{1}{2}}.$$

This shows that for any $\epsilon > 0$,

$$\|\Delta \tilde{P}_2\|_{L^2(\bar{S})} + \|\nabla \partial_z \tilde{P}_2\|_{L^2(\bar{S})} \leq \|F_0\|_{L^2(\bar{S})} + K_\eta \|\partial_z \tilde{P}_1\|_{L^2(\bar{S})} + \epsilon \|\partial_z \nabla \tilde{P}_2\|_{L^2(\bar{S})}. \quad (7.3)$$

Using the equation (7.2), we infer that

$$\begin{aligned} \|\partial_z^2 \tilde{P}_2\|_{L^2(\bar{S})} & \leq K_\eta (\|\Delta \tilde{P}_2\|_{L^2(\bar{S})} + \|\nabla \partial_z \tilde{P}_2\|_{L^2(\bar{S})}) \\ & \quad + \|\partial_z \tilde{P}_2\|_{L^2(\bar{S})}^{\frac{1}{2}} \|\partial_z^2 \tilde{P}_2\|_{L^2(\bar{S})}^{\frac{1}{2}} + \|F_0\|_{L^2(\bar{S})}, \end{aligned}$$

which along with (7.3) gives by taking ϵ small that

$$\begin{aligned} \|\nabla_{x,z}^2 \tilde{P}_2\|_{L^2(\Omega_t)} & \leq K_\eta (\|F_0\|_{L^2(\bar{S})} + \|\partial_z \tilde{P}_1\|_{L^2(\bar{S})}) \\ & \leq K_\eta (\|\eta\|_{H^{\frac{3}{2}}} + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} v\|_{L^2(\Omega_t)} + \|\partial_z \tilde{P}_1\|_{L^2(\bar{S})}). \end{aligned}$$

Then the lemma follows by using Lemma 7.1. \square

Next let us turn to H^1 estimate of $(\partial_t + v \cdot \nabla_{x,y})P \triangleq \mathcal{D}_t P$. Using the equation (1.1), a direct calculation gives

$$\begin{cases} \Delta_{x,y} \mathcal{D}_t P = \partial_k P \Delta_{x,y} v^k + G, \\ \mathcal{D}_t P|_{y=\eta} = 0, \quad \partial_y \mathcal{D}_t P|_{y=-1} = \nabla \cdot v^h + \partial_y v^h \cdot \nabla P, \end{cases} \quad (7.4)$$

where

$$\begin{aligned} G &= 4\delta^{ij} \partial_i v^k \partial_j \partial_k P - 2(\partial_i v^j)(\partial_j v^k) \partial_k v^i \\ &= 4\partial_k (\partial_i v^k \partial_i P) - 2(\partial_i v^j)(\partial_j v^k) \partial_k v^i. \end{aligned}$$

Recall $\omega_{i,k} = \partial_i v^k - \partial_k v^i$. We have

$$\partial_k P \cdot \Delta_{x,y} v^k = \partial_i (\partial_k P \omega_{i,k}) - \partial_i \partial_k P \omega_{i,k} = \partial_i (\partial_k P \omega_{i,k}).$$

Lemma 7.3. *Let $\mathcal{D}_t P$ be a solution of (7.4). It holds that*

$$\|\nabla_{x,y} \mathcal{D}_t P\|_{L^2(\Omega_t)} \leq K_\eta (1 + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2) (\|v\|_{H^1(\Omega_t)} + \|\eta\|_{H^{\frac{3}{2}}}).$$

Proof. We get by integration by parts that

$$\begin{aligned} & \int_{\Omega_t} |\nabla_{x,y} \mathcal{D}_t P|^2 dx dy + \int_{\mathbf{R}^d} \partial_y \mathcal{D}_t P \mathcal{D}_t P|_{y=-1} dx \\ &= - \int_{\Omega_t} (4(\partial_i v^k \partial_i P) \partial_k \mathcal{D}_t P + \partial_k P \omega_{i,k} \partial_i \mathcal{D}_t P + 2(\partial_i v^j)(\partial_j v^k) \partial_k v^i \mathcal{D}_t P) dx dy \\ & \quad + \int_{\mathbf{R}^d} (\partial_k P \omega_{d+1,k}) \mathcal{D}_t P + (\partial_i v^{d+1} \partial_i P) \mathcal{D}_t P dx|_{y=-1} \\ &\leq C (\|\nabla_{x,y} v\|_{L^2(\Omega_t)} + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} P_1\|_{L^2(\Omega_t)}) \|\nabla_{x,y} \mathcal{D}_t P\|_{L^2(\Omega_t)} \\ & \quad + C \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2 \|\nabla_{x,y} v\|_{L^2(\Omega_t)} \|\mathcal{D}_t P\|_{L^2(\Omega_t)} \\ & \quad + C \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} P_1(\cdot, -1)\|_{L^2(\mathbf{R}^d)} \|\mathcal{D}_t P(\cdot, -1)\|_{L^2} + \int_{\mathbf{R}^d} \nabla \cdot v^h \mathcal{D}_t P|_{y=-1} dx. \end{aligned}$$

Here we used $\operatorname{div} v = 0$. By the boundary condition of (7.4), we have

$$\begin{aligned} \int_{\mathbf{R}^d} \partial_y \mathcal{D}_t P \mathcal{D}_t P - \nabla \cdot v^h \mathcal{D}_t P|_{y=-1} dx &= \int_{\mathbf{R}^d} (\partial_y v^h \cdot \nabla P) \mathcal{D}_t P|_{y=-1} dx \\ &\leq \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} P_1(\cdot, -1)\|_{L^2} \|\mathcal{D}_t P(\cdot, -1)\|_{L^2} \\ &\leq K_\eta \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} P_1\|_{H^1(\Omega_t)} \|\mathcal{D}_t P\|_{H^1(\Omega_t)} \end{aligned}$$

Thanks to $\mathcal{D}_t P|_{y=\eta} = 0$ so that

$$\|\mathcal{D}_t P\|_{L^2(\Omega_t)} \leq K_\eta \|\mathcal{D}_t P\|_{H^1(\Omega_t)}.$$

we deduce that

$$\|\nabla_{x,y} \mathcal{D}_t P\|_{L^2(\Omega_t)} \leq K_\eta (1 + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2) (\|\nabla_{x,y} P_1\|_{H^1(\Omega_t)} + \|\nabla_{x,y} v\|_{L^2(\Omega_t)}),$$

which together with Lemma 7.2 concludes the lemma. \square

7.2. Hölder estimate of the pressure. In the sequel, we denote

$$A(t) \triangleq 1 + \|v(t)\|_{W^{1,\infty}(\Omega_t)} + \|v(t)\|_{H^1(\Omega_t)} + \|\eta(t)\|_{H^{\frac{3}{2}}}. \quad (7.5)$$

First, we give the Hölder estimate of the pressure.

Lemma 7.4. *Let $I_0 = [-\frac{3}{4}, 0]$ and P be a solution of (7.1). Then it holds that*

$$\|\nabla_{x,z}\tilde{P}\|_{\tilde{L}_z^\infty(I_0;C^{\frac{1}{2}})} \leq K_\eta A(t)^2.$$

Proof. Apply Proposition 4.16 with $F_0 = \alpha\widetilde{\partial_i v^j \partial_j v^i}$ and $G_0 = 0$ to obtain

$$\|\nabla_{x,z}\tilde{P}\|_{\tilde{L}_z^\infty(I_0;C^{\frac{1}{2}})} \leq K_\eta (\|F_0\|_{\tilde{L}_z^2(I;C^0)} + \|\nabla_{x,z}\tilde{P}\|_{\tilde{L}_z^\infty(I_0;C^{-\delta})} + \|\nabla_{x,z}\tilde{P}\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\delta}{2}})}),$$

where $I_1 = [-\frac{7}{8}, 0]$ and $\delta > 0$ is taken so that $-\delta + \frac{d}{2} \leq -\frac{1}{2}$. It is obvious that

$$\|F_0\|_{L_z^2(I;C^0)} \leq K_\eta \|\nabla_{x,y}v\|_{L^\infty(\Omega_t)}^2.$$

It follows from Lemma 4.7 and Lemma 7.1 that

$$\begin{aligned} \|\nabla_{x,z}\tilde{P}\|_{\tilde{L}_z^\infty(I_0;C^{-\delta})} &\leq \|\nabla_{x,z}\tilde{P}_1\|_{\tilde{L}_z^\infty(I_0;C^{-\delta})} + \|\nabla_{x,z}\rho\delta\|_{\tilde{L}_z^\infty(I_0;C^{-\delta})} \\ &\leq C\|\nabla_{x,z}\tilde{P}_1\|_{L_z^\infty(I_0;H^{-\frac{1}{2}})} + C\|\eta\|_{L^\infty} \\ &\leq K_\eta (\|\nabla_{x,y}P_1\|_{L^2(\Omega_t)} + \|F_0\|_{L_z^2(I;L^2)}) + C\|\eta\|_{L^\infty} \\ &\leq K_\eta (\|\nabla_{x,y}v\|_{L^\infty(\Omega_t)}\|v\|_{H^1(\Omega_t)} + \|\eta\|_{H^{\frac{1}{2}}} + 1). \end{aligned}$$

Take $c_1, c_2 \in (0, 1)$ depending on K_η, a such that $\rho_\delta(x, I_1 \setminus I_0) \in \mathcal{S}_1 = \{(x, y) : y \in [-1 + c_1 h_0, \eta(x) - c_2 h_0]\}$. Let δ_1 be as in Proposition 4.18. Then for p big enough depending on ε and d , we have

$$\begin{aligned} \|\nabla_{x,z}\tilde{P}\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\delta}{2}})} &\leq C(\delta_1)\|\nabla_{x,z}\tilde{P}_1\|_{L_z^2(I;L^2)} + C\|\eta\|_{W^{1,\infty}} \\ &\quad + \sup_{X_0 \in \mathcal{S}_1} \|\nabla_{x,y}P\|_{L^p(B_{\delta_1}(X_0))}, \end{aligned}$$

which together with Proposition 4.18 and Lemma 7.1 implies that

$$\begin{aligned} \|\nabla_{x,z}\tilde{P}\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\delta}{2}})} &\leq K_\eta (\|\nabla_{x,y}v\|_{L^\infty(\Omega_t)}^2 + \|\nabla_{x,y}P_1\|_{L^2(\Omega_t)}) + K_\eta \\ &\leq K_\eta A(t)^2. \end{aligned} \quad (7.6)$$

Putting the above estimates together concludes the lemma. \square

Next, we give the estimate of $\nabla_{x,z}\widetilde{\mathcal{D}_t P}$ in Besov space:

Lemma 7.5. *Let $I_0 = [-\frac{3}{4}, 0]$ and $\mathcal{D}_t P$ be a solution of (7.4). Then it holds that*

$$\|\nabla_{x,z}\widetilde{\mathcal{D}_t P}\|_{\tilde{L}_z^\infty(I_0;B_{\infty,1}^0)} \leq K_\eta (1 + \|\nabla_{x,z}\tilde{v}\|_{L_z^\infty(I;B_{\infty,1}^0)})A(t)^2.$$

Proof. We denote

$$\begin{aligned} F_0 &= \sum_{i=1}^d \partial_i (\alpha (\widetilde{\partial_k P \omega_{i,k}} + 4\widetilde{\partial_k v^i \partial_k P})) - 2\alpha (\widetilde{\partial_i v^j})(\widetilde{\partial_j v^k})\widetilde{\partial_k v^i} \\ &\quad + \left(\sum_{i=1}^d (\partial_z \alpha_i - \partial_i \alpha) (\widetilde{\partial_k P \omega_{i,k}} + 4\widetilde{\partial_k v^i \partial_k P}) \right. \\ &\quad \left. - \partial_z \alpha_{d+1} (\widetilde{\partial_k P \omega_{d+1,k}} + 4\widetilde{\partial_k v^{d+1} \partial_k P}) \right) \triangleq F_0^1 + F_0^2 + F_0^3, \end{aligned} \quad (7.7)$$

$$G_0 = - \sum_{i=1}^d \alpha_i (\widetilde{\partial_k P \omega_{i,k}} + 4 \widetilde{\partial_k v^i \partial_k P}) + \alpha_{d+1} (\widetilde{\partial_k P \omega_{d+1,k}} + 4 \widetilde{\partial_k v^{d+1} \partial_k P}), \quad (7.8)$$

where $\alpha_1 = \alpha \frac{\partial_i \rho_\delta}{\partial_z \rho_\delta}$ for $i = 1, \dots, d$ and $\alpha_{d+1} = \frac{\alpha}{\partial_z \rho_\delta}$. Then $\mathcal{D}_t P$ satisfies

$$\partial_z^2 \mathcal{D}_t P + \alpha \Delta \mathcal{D}_t P + \beta \cdot \nabla \partial_z \mathcal{D}_t P - \gamma \partial_z \mathcal{D}_t P = F_0 + \partial_z G_0.$$

We apply Proposition 4.16 to obtain

$$\begin{aligned} \|\nabla_{x,z} \widetilde{\mathcal{D}_t P}\|_{\tilde{L}_z^\infty(I_0; B_{\infty,1}^0)} &\leq K_\eta (\|F_0\|_{Y_1^0(I)} + \|G_0\|_{\tilde{L}_z^\infty(I_0; B_{\infty,1}^0)} + \|\nabla_{x,z} \widetilde{\mathcal{D}_t P}\|_{\tilde{L}_z^\infty(I_0; C^{-\delta})} \\ &\quad + \|\nabla_{x,z} \widetilde{\mathcal{D}_t P}\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})}), \end{aligned}$$

where $I_1 = [-\frac{7}{8}, 0]$. By Lemma 7.6 and Lemma 7.4, we have

$$\begin{aligned} &\|F_0\|_{Y_1^0(I)} + \|G_0\|_{\tilde{L}_z^\infty(I_0; B_{\infty,1}^0)} \\ &\leq K_\eta \|\nabla_{x,z} \tilde{v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)} (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2 + \|\nabla_{x,z} \tilde{P}\|_{\tilde{L}_z^\infty(I; C^{\frac{1}{2}})}) \\ &\leq K_\eta \|\nabla_{x,z} \tilde{v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)} A(t)^2. \end{aligned} \quad (7.9)$$

We get by Lemma 7.3 and Lemma 4.7 that

$$\begin{aligned} \|\nabla \widetilde{\mathcal{D}_t P}\|_{\tilde{L}_z^\infty(I_0; C^{-\delta})} &\leq C \|\nabla \widetilde{\mathcal{D}_t P}\|_{L_z^\infty(I_0; H^{-\frac{1}{2}})} \leq C \|\nabla_{x,y} \mathcal{D}_t P\|_{L^2(\Omega_t)} \\ &\leq K_\eta (1 + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}) A(t)^2. \end{aligned}$$

On the other hand, for $z \in [-1, 0]$,

$$\begin{aligned} \partial_z \widetilde{\mathcal{D}_t P}(x, z) &= \int_{-1}^z \partial_z^2 \widetilde{\mathcal{D}_t P}(x, z') dz' \\ &= G(z) - G(-1) + \int_{-1}^z (F_0 - \alpha \Delta \widetilde{\mathcal{D}_t P} + \beta \nabla \partial_z \widetilde{\mathcal{D}_t P} - \gamma \partial_z \widetilde{\mathcal{D}_t P}) dz'. \end{aligned}$$

From Lemma 4.8 in [32] and Lemma 7.3, we know that

$$\begin{aligned} \|\alpha \Delta \widetilde{\mathcal{D}_t P} - \beta \nabla \partial_z \widetilde{\mathcal{D}_t P} + \gamma \partial_z \widetilde{\mathcal{D}_t P}\|_{L_z^1(-1, 0; H^{-1})} &\leq K_\eta \|\nabla_{x,z} \widetilde{\mathcal{D}_t P}\|_{L^2(\overline{\mathcal{S}})} \\ &\leq K_\eta (1 + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}) A(t), \end{aligned}$$

which along with (7.9) implies

$$\|\partial_z \widetilde{\mathcal{D}_t P}\|_{\tilde{L}_z^\infty(I_0; C^{-\delta})} \leq K_\eta (1 + \|\nabla_{x,z} \tilde{v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)}) A(t)^2.$$

A similar argument leading to (7.6) yields

$$\begin{aligned} &\|\nabla_{x,z} \widetilde{\mathcal{D}_t P}\|_{\tilde{L}_z^2(I_1 \setminus I_0; C^{-\frac{\epsilon}{2}})} \\ &\leq K_\eta (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} P\|_{L^\infty(\Omega_t)} + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^3 + \|\nabla_{x,y} \mathcal{D}_t P\|_{L^2(\Omega_t)}) \\ &\leq K_\eta (1 + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}) A(t)^2. \end{aligned}$$

Putting the above estimates together concludes the lemma. \square

Next, we give the estimates of F_0 and G_0 .

Lemma 7.6. *Let F_0 and G_0 be given by (7.7) and (7.8) respectively. Then we have*

$$\begin{aligned} & \|F_0\|_{Y_1^0(I)} + \|G_0\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)} \\ & \leq K_\eta \|\nabla_{x,z} \tilde{v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)} (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2 + \|\nabla_{x,z} \tilde{P}\|_{L_z^\infty(I; C^{\frac{1}{2}})}). \end{aligned}$$

Proof. We can deduce from Lemma 2.17, Lemma 2.18 that for $\delta < 0$,

$$\begin{aligned} \|fg\|_{\tilde{L}_z^p(I; B_{\infty,1}^\delta)} & \leq C \|f\|_{\tilde{L}_z^p(I; B_{\infty,1}^0)} \|g\|_{L^\infty(\bar{S})}, \\ \|fg\|_{\tilde{L}_z^p(I; B_{\infty,1}^0)} & \leq C \|f\|_{\tilde{L}_z^p(I; B_{\infty,1}^0)} \|g\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)}, \end{aligned}$$

which together with Lemma 4.4 imply that

$$\begin{aligned} \|G_0\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)} & \leq K_\eta \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,z} \tilde{P}\|_{\tilde{L}_z^\infty(I; C^{\frac{1}{2}})}, \\ \|F_0^1\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^{-1})} & \leq K_\eta \|\nabla_{x,z} \tilde{v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)} \|\nabla_{x,z} \tilde{P}\|_{\tilde{L}_z^\infty(I; C^{\frac{1}{2}})}, \\ \|F_0^2\|_{\tilde{L}_z^2(I_0; B_{\infty,1}^{-\frac{1}{2}})} & \leq K_\eta \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2 \|\nabla_{x,z} \tilde{v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)}, \\ \|F_0^3\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})} & \leq K_\eta \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,z} \tilde{P}\|_{\tilde{L}_z^\infty(I; C^{\frac{1}{2}})}. \end{aligned}$$

Here we also used the fact that

$$\|\widetilde{\nabla_{x,y} v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)} \leq K_\eta \|\nabla_{x,z} \tilde{v}\|_{\tilde{L}_z^\infty(I; B_{\infty,1}^0)}.$$

This gives the lemma by the definition of $Y_1^0(I)$. \square

7.3. Sobolev estimate of the pressure.

Lemma 7.7. *Let $I_0 = [-\frac{1}{2}, 0]$ and P be a solution of (7.1). Then it holds that*

$$\begin{aligned} \|\nabla_{x,z} \tilde{P}_1\|_{X^{s-\frac{1}{2}}(I_0)} & \leq K_\eta A(t)^2 (\|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-1})} + \|\eta\|_{H^{s+\frac{1}{2}}}), \\ \|\nabla_{x,y} P_1(\cdot, -1)\|_{H^s} & \leq K_\eta A(t)^2 (\|\nabla_{x,z} \tilde{v}\|_{H^{s-\frac{1}{2}}(\bar{S})} + \|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

Proof. Apply Proposition 4.12 with $F_0 = \alpha(\widetilde{\partial_i v^j \partial_j v^i})$ and $f = 0$ to obtain

$$\begin{aligned} \|\nabla_{x,z} \tilde{P}_1\|_{X^{s-\frac{1}{2}}(I_0)} & \leq K_\eta (\|\nabla_{x,z} \tilde{P}_1\|_{L^2(\bar{S})} + \|F_0\|_{Y^{s-\frac{1}{2}}(I)} \\ & \quad + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z} \tilde{P}_1\|_{L^\infty(\mathbf{R}^d \times I_1)}) \end{aligned}$$

with $I_1 = [-\frac{3}{4}, 0]$. By Lemma 2.11 and Lemma 4.4, we have

$$\begin{aligned} \|F_0\|_{Y^{s-\frac{1}{2}}(I)} & \leq \|\widetilde{\partial_i v^j \partial_j v^i}\|_{L_z^2(I; H^{s-1})} + \|(\alpha - 1)(\widetilde{\partial_i v^j \partial_j v^i})\|_{L_z^2(I; H^{s-1})} \\ & \leq K_\eta \|\widetilde{\partial_i v^j}\|_{L^\infty(\bar{S})} \|\widetilde{\partial_j v^i}\|_{L_z^2(I; H^{s-1})} + C \|\widetilde{\partial_i v^j}\|_{L^\infty(\bar{S})}^2 \|\alpha - 1\|_{L_z^2(I; H^{s-1})} \\ & \leq K_\eta (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-1})} + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2 \|\eta\|_{H^{s-\frac{1}{2}}}), \end{aligned}$$

which along with Lemma 7.1 and Lemma 7.4 gives the first inequality.

In fact, the first inequality also holds with $[a, 0]$ for $a > -1$ instead of I_0 . This implies there exists $y_0 \in (-1, -1 + h_0)$ so that

$$\|P_1(\cdot, y_0)\|_{H^s} \leq K_\eta A(t)^2 (\|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-1})} + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

Let $\mathcal{S}_1 = \mathbf{R}^d \times [-1, y_1]$ where $y_1 < y_0$. Then the standard elliptic estimate ensures that

$$\begin{aligned} \|\nabla_{x,y} P_1\|_{H^{s+\frac{1}{2}}(\mathcal{S}_1)} &\leq C(\|P_1(\cdot, y_0)\|_{H^s} + \|\nabla_{x,y} v \nabla_{x,y} v\|_{H^{s-\frac{1}{2}}(\mathcal{S}_1)}) \\ &\leq K_\eta A(t)^2 (\|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-1})} + \|\nabla_{x,y} v\|_{H^{s-\frac{1}{2}}(\mathcal{S}_1)} + \|\eta\|_{H^{s+\frac{1}{2}}}), \end{aligned}$$

which together with the fact that

$$\|\nabla_{x,y} v\|_{H^{s-\frac{1}{2}}(\mathcal{S}_1)} \leq K_\eta (\|\nabla_{x,z} \tilde{v}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} + \|\eta\|_{H^{s+\frac{1}{2}}}),$$

implies the second inequality by the trace theorem. \square

7.4. Estimate of a . Recall that $a(t, x) = -(\partial_y P)(t, x, \eta(t, x))$.

Lemma 7.8. *It holds that*

$$\begin{aligned} \|a\|_{C^{\frac{1}{2}}} &\leq K_\eta A(t)^2, \\ \|D_t a\|_{B_{\infty,1}^0} &\leq K_\eta (1 + \|\tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^1)}) A(t)^2. \end{aligned}$$

Proof. The first inequality is a direct consequence of Lemma 7.4 and Lemma 4.5 by using $\partial_y P = \frac{\partial_z \tilde{P}}{\partial_z \rho_\delta}$. To show the second inequality, we write

$$\begin{aligned} \partial_t a + T_V \cdot \nabla a &= \partial_y (\partial_t + v \cdot \nabla_{x,y} P) - \partial_y v \cdot \nabla_{x,y} P|_{y=\eta} + (T_V - V) \cdot \nabla a \\ &= \frac{\partial_z \widetilde{\mathcal{D}_t P}}{\partial_z \rho_\delta} - \frac{\partial_z \tilde{v}}{\partial_z \rho} \cdot \left(\nabla \tilde{P} - \frac{\nabla \rho_\delta}{\partial_z \rho_\delta} \partial_z \tilde{P}, \frac{\partial_z \tilde{P}}{\partial_z \rho} \right) \Big|_{z=0} + (T_V - V) \cdot \nabla a \\ &\triangleq I_1 + I_2 + I_3. \end{aligned}$$

Using Lemma 2.10, Lemma 2.12 and Lemma 4.5, it is easy to show that

$$\begin{aligned} \|I_1\|_{B_{\infty,1}^0} &\leq K_\eta \|\partial_z \widetilde{\mathcal{D}_t P}\|_{L_z^\infty(I; B_{\infty,1}^0)}, \\ \|I_2\|_{B_{\infty,1}^0} &\leq K_\eta \|\nabla_{x,z} \tilde{P}\|_{L_z^\infty(I; C^{\frac{1}{2}})} \|\nabla_{x,z} \tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^0)}, \\ \|I_3\|_{B_{\infty,1}^0} &\leq C \|V\|_{B_{\infty,1}^1} \|a\|_{C^{\frac{1}{2}}} \leq C \|\tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^1)} \|a\|_{C^{\frac{1}{2}}}, \end{aligned}$$

which along with Lemma 7.4 and Lemma 7.5 lead to the second inequality. \square

Lemma 7.9. *It holds that*

$$\|a - 1\|_{H^{s-\frac{1}{2}}} \leq K_\eta A(t)^2 (\|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-1})} + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

Proof. Note that $a - 1 = \frac{1}{\partial_z \rho_\delta} \partial_z \tilde{P}_1 \Big|_{z=0}$. Then we deduce from Lemma 2.11, Lemma 2.12 and Lemma 4.5 that

$$\|a - 1\|_{H^{s-\frac{1}{2}}} \leq K_\eta (\|\partial_z \tilde{P}_1\|_{L_z^\infty([-\frac{1}{2}, 0]; H^{s-\frac{1}{2}})} + \|\eta\|_{H^{s+\frac{1}{2}}}),$$

which together with Lemma 7.7 yields the result. \square

8. ESTIMATE OF THE VELOCITY

The velocity v satisfies

$$\begin{cases} \Delta_{x,y} v = -\nabla_{x,y} \times \omega & \text{in } \Omega_t, \\ v|_{y=\eta} = (V, B), \quad v|_{y=-1} = (V_b, 0). \end{cases}$$

Here ω is the vorticity.

8.1. Sobolev estimate of the velocity. The following H^1 estimate is classical:

$$\|v\|_{H^1(\Omega_t)} \leq K_\eta (\|\omega\|_{L^2(\Omega_t)} + \|(V, B, V_b)\|_{H^{\frac{1}{2}}}). \quad (8.1)$$

This together with Proposition 4.12 and Proposition 4.14 ensures that

Lemma 8.1. *Let $I_0 = [a, 0]$ for $a \in (-1, 0)$. It holds that*

$$\|\nabla_{x,z}\tilde{v}\|_{X^{s-1}(I_0)} \leq K_\eta (\|(V, B, V_b)\|_{H^s} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{S})} + \|\nabla_{x,y}v\|_{L^\infty(\Omega_t)}\|\eta\|_{H^{s+\frac{1}{2}}}).$$

If $s - \frac{1}{2}$ is an integer, we have

$$\|\nabla_{x,z}\tilde{v}\|_{H^{s-\frac{1}{2}}(\overline{S})} \leq K_\eta (\|(V, B, V_b)\|_{H^s} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{S})} + \|\nabla_{x,y}v\|_{L^\infty(\Omega_t)}\|\eta\|_{H^{s+\frac{1}{2}}}).$$

8.2. The estimate of the irrotational part. The irrotational part v_{ir} of the velocity is defined by

$$\begin{cases} \Delta_{x,y}v_{ir} = 0 & \text{in } \Omega_t, \\ v_{ir}|_{y=\eta} = (V, B), & v_{ir}|_{y=-1} = 0. \end{cases}$$

Lemma 8.2. *Let $I_0 = [-\frac{3}{4}, 0]$. It holds that*

$$\begin{aligned} \|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^\infty(I_0;C^0)} &\leq K_\eta A(t), \\ \|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^\infty(I_0;B_{\infty,1}^0)} &\leq K_\eta (\|\tilde{v}\|_{L^\infty(I;B_{\infty,1}^1)} + \|v\|_{H^1(\Omega_t)}), \end{aligned}$$

where $A(t)$ is given by (7.5).

Proof. Let $I_1 = [-\frac{7}{8}, 0]$. It follows from Proposition 4.16 that

$$\|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^\infty(I_0;C^0)} \leq K_\eta (\|(V, B)\|_{C^1} + \|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^\infty(I_0;C^{-\delta})} + \|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\epsilon}{2}})})$$

for any $\delta > 0$. Take $\delta > 0$ so that $-\delta + \frac{d}{2} \leq -\frac{1}{2}$. Then we get by Lemma 4.7 that

$$\begin{aligned} \|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^\infty(I_0;C^{-\delta})} &\leq C \|\nabla_{x,z}\tilde{v}_{ir}\|_{L^\infty(I_0;H^{-\frac{1}{2}})} \\ &\leq K_\eta \|\nabla_{x,y}v_{ir}\|_{L^2(\Omega_t)} \leq K_\eta \|(V, B)\|_{H^{\frac{1}{2}}}. \end{aligned}$$

While, Proposition 4.18 implies that

$$\|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^2(I_1 \setminus I_0;C^{-\frac{\epsilon}{2}})} \leq K_\eta \|\nabla_{x,y}v_{ir}\|_{L^2(\Omega_t)} \leq K_\eta \|(V, B)\|_{H^{\frac{1}{2}}}.$$

This shows that

$$\|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^\infty(I_0;C^0)} \leq K_\eta (\|(V, B)\|_{C^1} + \|(V, B)\|_{H^{\frac{1}{2}}}) \leq K_\eta A(t).$$

Similar argument leads to

$$\begin{aligned} \|\nabla_{x,z}\tilde{v}_{ir}\|_{\tilde{L}_z^\infty(I_0;B_{\infty,1}^0)} &\leq K_\eta (\|(V, B)\|_{B_{\infty,1}^1} + \|(V, B)\|_{H^{\frac{1}{2}}}) \\ &\leq K_\eta (\|\tilde{v}\|_{L^\infty(I;B_{\infty,1}^1)} + \|v\|_{H^1(\Omega_t)}). \end{aligned}$$

This finishes the proof. \square

With Lemma 8.2, we deduce from Proposition 5.4 and Proposition 5.5 that

Proposition 8.3. *It holds that*

$$\begin{aligned} \|R(\eta)(V, B)\|_{C^{\frac{1}{2}}} &\leq K_\eta A(t), \\ \|R(\eta)(V, B)\|_{H^{s-1}} &\leq K_\eta (\|(V, B)\|_{H^{s-\frac{1}{2}}} + A(t)\|\eta\|_{H^{s+\frac{1}{2}}}), \\ \|R(\eta)(V, B)\|_{H^{s-\frac{1}{2}}} &\leq K_\eta (\|(V, B)\|_{H^s} + (\|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)} + A(t))\|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

8.3. The estimate of the rotational part. The rotational part v_ω of the velocity is defined by

$$\begin{cases} \Delta_{x,y} v_\omega = -\nabla_{x,y} \times \omega & \text{in } \Omega_t, \\ v_\omega|_{y=\eta} = 0, \quad v_\omega|_{y=-1} = (V_b, 0). \end{cases}$$

Lemma 8.4. *Let $I_0 = [-\frac{3}{4}, 0]$. It holds that*

$$\begin{aligned} \|\nabla_{x,z} \widetilde{v_\omega}\|_{\tilde{L}_z^\infty(I_0; C^0)} &\leq K_\eta A(t), \\ \|\nabla_{x,z} \widetilde{v_\omega}\|_{\tilde{L}_z^\infty(I_0; B_{\infty,1}^0)} &\leq K_\eta (\|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)} + \|v\|_{H^1(\Omega_t)}), \\ \|\nabla_{x,z} \widetilde{v_\omega}\|_{X^{s-1}([-\frac{1}{2}, 0])} &\leq K_\eta (\|V_b\|_{H^{\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\mathcal{S})} + A(t)\|\eta\|_{H^s}). \end{aligned}$$

Proof. The first two inequalities follows from Lemma 8.2 and the fact $v_\omega = v - v_{ir}$. Then we get by Proposition 4.12 that

$$\|\nabla_{x,z} \widetilde{v_\omega}\|_{X^{s-1}([-\frac{1}{2}, 0])} \leq K_\eta (\|\nabla_{x,y} v_\omega\|_{L^2(\Omega_t)} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\mathcal{S})} + \|\nabla_{x,y} \widetilde{v_\omega}\|_{\tilde{L}_z^\infty(I_0; C^0)} \|\eta\|_{H^s}),$$

which along with the first inequality yields the third inequality. \square

Next we show that $(\partial_t + v \cdot \nabla_{x,y})v_\omega \triangleq \mathcal{D}_t v_\omega$ has the similar estimates. It is easy to verify that $\mathcal{D}_t v_\omega$ satisfies

$$\begin{cases} \Delta_{x,y} \mathcal{D}_t v_\omega = G_\omega & \text{in } \Omega_t, \\ \mathcal{D}_t v_\omega|_{y=\eta} = 0, \quad \mathcal{D}_t v_\omega|_{y=-1} = (\dot{V}_b, 0). \end{cases}$$

where $\dot{V}_b = (\partial_t + V_b \cdot \nabla)V_b$ and

$$G_\omega = -\nabla_{x,y} \times \mathcal{D}_t \omega + \nabla_{x,y} \cdot (\omega \cdot \nabla_{x,y} \times v) + \nabla \times (\omega \cdot \nabla v_\omega) + 2\partial_i (\nabla_k v^i \cdot \nabla_k v_\omega).$$

It is easy to find that G_ω can be rewritten $\tilde{G}_\omega = \tilde{G}_\omega^0 + \nabla_z \tilde{G}_\omega^1$ where G_ω^1 satisfies that (by Lemma 2.13 and Lemma 4.3),

$$\begin{aligned} &\|\tilde{G}_\omega^0\|_{L^2(I_0; H^{s-\frac{3}{2}})} + \|\tilde{G}_\omega^1\|_{L^2(I_0; H^{s-\frac{1}{2}})} \\ &\leq K_\eta (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2 + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} v_\omega\|_{L^\infty(\Omega_t)}) \|\eta\|_{H^{s+\frac{1}{2}}} \\ &\quad + K_\eta (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} + \|\nabla_{x,y} v_\omega\|_{L^\infty(\Omega_t)}) \|\nabla_{x,z} \tilde{v}\|_{L^2(I_0; H^{s-\frac{1}{2}})} \\ &\quad + K_\eta \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,z} \widetilde{v_\omega}\|_{L^2(I_0; H^{s-\frac{1}{2}})}. \end{aligned}$$

Lemma 8.5. *It holds that*

$$\begin{aligned} \|\nabla_{x,z} \widetilde{\mathcal{D}_t v_\omega}\|_{\tilde{L}^\infty([-\frac{1}{2}, 0]; C^0)} &\leq K_\eta (1 + \|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)}) A(t)^2, \\ \|\nabla_{x,z} \widetilde{\mathcal{D}_t v_\omega}\|_{X^{s-1}([-\frac{1}{2}, 0])} &\leq K_\eta (1 + \|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)}) A(t)^2 \\ &\quad \times (\|\eta\|_{H^{s+\frac{1}{2}}} + \|(V, B, V_b)\|_{H^s} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}}). \end{aligned}$$

Proof. Thanks to $\mathcal{D}_t\omega = \omega \cdot \nabla v$, we get

$$\|\mathcal{D}_t\omega\|_{L^2(\Omega_t)} \leq \|\nabla_{x,y}v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y}v\|_{L^2(\Omega_t)},$$

and by (6.3) and Lemma 7.2, we have

$$\|\dot{V}_b\|_{H^{\frac{1}{2}}} \leq K_\eta \|\nabla_{x,y}P\|_{H^1(\Omega_t)} \leq K_\eta A(t)^2.$$

Then it is easy to show that

$$\begin{aligned} \|\nabla_{x,y}\mathcal{D}_tv_\omega\|_{L^2(\Omega_t)} &\leq K_\eta (\|\dot{V}_b\|_{H^{\frac{1}{2}}} + \|\nabla_{x,y}v\|_{L^\infty(\Omega_t)} (\|\nabla_{x,y}v\|_{L^2(\Omega_t)} + \|\nabla_{x,y}v_\omega\|_{L^2(\Omega_t)})) \\ &\leq K_\eta A(t)^2. \end{aligned}$$

Then a similar argument leading to Lemma 7.5 ensures that

$$\begin{aligned} \|\nabla_{x,z}\widetilde{\mathcal{D}_tv_\omega}\|_{L^\infty([- \frac{3}{4}, 0]; C^0)} &\leq K_\eta (A(t)^2 + \|\nabla_{x,y}v_\omega\|_{L^\infty(\Omega_t)} A(t)) \\ &\leq K_\eta (1 + \|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)}) A(t)^2. \end{aligned}$$

Let $I_0 = [-\frac{1}{2}, 0]$ and $I_1 = [-\frac{3}{4}, 0]$. Then Proposition 4.12 together with Lemma 8.1 and Lemma 8.4 implies that

$$\begin{aligned} \|\nabla_{x,z}\widetilde{\mathcal{D}_tv_\omega}\|_{X^{s-1}(I_0)} &\leq K_\eta (\|\nabla_{x,y}\mathcal{D}_tv_\omega\|_{L^2(\Omega_t)} + \|G_\omega^0\|_{L^2(I_0; H^{s-\frac{3}{2}})} + \|G_\omega^1\|_{L^2(I_0; H^{s-\frac{1}{2}})} \\ &\quad + \|\nabla_{x,z}\widetilde{\mathcal{D}_tv_\omega}\|_{L^\infty(I_1; C^0)} \|\eta\|_{H^{s+\frac{1}{2}}}) \\ &\leq K_\eta (1 + \|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)}) A(t)^2 \\ &\quad \times (\|\eta\|_{H^{s+\frac{1}{2}}} + \|(V, B, V_b)\|_{H^s} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}}). \end{aligned}$$

The proof is finished. \square

9. PROOF OF BREAK-DOWN CRITERION

In this section, we assume that (η, v) is a solution of the system (1.1)–(1.5) obtained in Theorem 1.1 in $\Omega_t = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : -1 < y < \eta(t, x)\}$ for $t \in [0, T]$. We denote

$$\begin{aligned} A(t) &\triangleq 1 + \|v(t)\|_{W^{1,\infty}(\Omega_t)} + \|v(t)\|_{H^1(\Omega_t)} + \|\eta(t)\|_{H^{\frac{3}{2}}} + \frac{1}{c_0}, \\ \mathcal{B}(t) &\triangleq 1 + \|\tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^1)}. \end{aligned}$$

9.1. The H^1 energy estimate. We have the following basic energy law for the system (1.1)–(1.5).

Lemma 9.1. *For any $t \in [0, T]$, there holds*

$$E(t) = E(0), \quad E(t) \triangleq \|v(t)\|_{L^2(\Omega_t)}^2 + \|\eta(t)\|_{L^2}^2.$$

Proof. By (1.1), (1.4) and integration by parts, we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} |v(t, x, y)|^2 dx dy &= \int_{\mathbf{R}^d} \partial_t \eta |v|^2 dx + 2 \int_{\Omega_t} \partial_t v \cdot v dx dy \\ &= \int_{\mathbf{R}^d} \partial_t \eta |v|^2 dx - 2 \int_{\Omega_t} (v \cdot \nabla_{x,y} v + \nabla(P + y)) \cdot v dx dy \\ &= \int_{\mathbf{R}^d} \partial_t \eta |v|^2 dx - \int_{\mathbf{R}^d} v \cdot \mathbf{n}_+ (|v|^2 + 2\eta) \sqrt{1 + |\nabla \eta|^2} dx \end{aligned}$$

$$= -2 \int_{\mathbf{R}^d} \partial_t \eta(t, x) \eta(t, x) dx = -\frac{d}{dt} \int_{\mathbf{R}^d} |\eta(t, x)|^2 dx.$$

This shows that

$$\frac{d}{dt} \left(\int_{\Omega_t} |v(t, x, y)|^2 dx dy + \int_{\mathbf{R}^d} |\eta(t, x)|^2 dx \right) = 0.$$

Hence, $E(t) = E(0)$ for $t \in [0, T]$. \square

Lemma 9.2. *It holds that for any $t \in [0, T]$,*

$$\frac{d}{dt} \|\nabla_{x,y} v(t)\|_{L^2(\Omega_t)} \leq K_\eta (\|v(t)\|_{W^{1,\infty}(\Omega_t)} \|v(t)\|_{H^1(\Omega_t)} + \|\eta(t)\|_{H^{\frac{3}{2}}}).$$

Proof. Similar to the proof of Lemma 9.1, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega_t} |\nabla_{x,y} v(t, x, y)|^2 dx dy \\ &= \int_{\mathbf{R}^d} \partial_t \eta |\nabla_{x,y} v|^2 dx + 2 \int_{\Omega_t} \partial_t \nabla_{x,y} v \cdot \nabla_{x,y} v dx dy \\ &= \int_{\mathbf{R}^d} \partial_t \eta |\nabla_{x,y} v|^2 dx - \int_{\Omega_t} v \cdot \nabla_{x,y} |\nabla_{x,y} v|^2 dx dy \\ &\quad - 2 \int_{\Omega_t} (\nabla_{x,y} v \cdot \nabla_{x,y} v + \nabla_{x,y}^2 (P + y)) \cdot \nabla_{x,y} v dx dy \\ &= -2 \int_{\Omega_t} (\nabla_{x,y} v \cdot \nabla_{x,y} v + \nabla_{x,y}^2 (P + y)) \cdot \nabla_{x,y} v dx dy \\ &\leq 2 \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,y} v\|_{L^2(\Omega_t)}^2 + \|\nabla_{x,y}^2 (P + y)\|_{L^2(\Omega_t)} \|\nabla_{x,y} v\|_{L^2(\Omega_t)}, \end{aligned}$$

which along with Lemma 7.2 yields the result. \square

9.2. Energy estimate of the trace of the velocity and the free surface. Let us first present the lower order energy estimate.

Proposition 9.3. *It holds that*

$$\begin{aligned} & \frac{d}{dt} (\|(V, B)\|_{H^{s-\frac{1}{2}}}^2 + \|V_b\|_{H^s}^2 + \|\eta\|_{H^s}^2) \\ & \leq K_\eta A(t)^2 \left(\|(V, B, V_b)\|_{H^s}^2 + \|\nabla_{x,z} \tilde{v}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})}^2 + \|\eta\|_{H^{s+\frac{1}{2}}}^2 \right). \end{aligned}$$

Proof. Recall that $\eta(t, x)$ satisfies

$$\partial_t \eta + V \cdot \nabla \eta = B.$$

Make H^s energy estimate to obtain

$$\frac{1}{2} \frac{d}{dt} \|\eta(t)\|_{H^s}^2 \leq -\langle \langle D \rangle^s (V \cdot \nabla \eta), \langle D \rangle^s \eta \rangle + \|B\|_{H^s} \|\eta\|_{H^s}.$$

We write

$$\begin{aligned} \langle \langle D \rangle^s (V \cdot \nabla \eta), \langle D \rangle^s \eta \rangle &= \langle \langle D \rangle^s (T_V \cdot \nabla \eta), \langle D \rangle^s \eta \rangle + \langle \langle D \rangle^s ((V - T_V) \cdot \nabla \eta), \langle D \rangle^s \eta \rangle \\ &= \langle [\langle D \rangle^s, T_V] \cdot \nabla \eta, \langle D \rangle^s \eta \rangle - \langle T_{\nabla \cdot V} \langle D \rangle^s \eta, \langle D \rangle^s \eta \rangle \\ &\quad + \langle \langle D \rangle^s ((V - T_V) \cdot \nabla \eta), \langle D \rangle^s \eta \rangle. \end{aligned}$$

Then we deduce from Lemma 2.19 and Lemma 2.10 that

$$\langle \langle D \rangle^s (V \cdot \nabla \eta), \langle D \rangle^s \eta \rangle \leq C \|V\|_{W^{1,\infty}} \|\eta\|_{H^s}^2 + K_\eta \|V\|_{H^s} \|\eta\|_{H^s}.$$

This shows that

$$\frac{d}{dt} \|\eta(t)\|_{H^s}^2 \leq K_\eta A(t) (\|(V, B)\|_{H^s}^2 + \|\eta\|_{H^s}^2). \quad (9.1)$$

Recall that (V, B, V_b) satisfies

$$\begin{aligned} (\partial_t + V \cdot \nabla) B &= a - 1, \\ (\partial_t + V \cdot \nabla) V + a\zeta &= 0, \\ (\partial_t + V_b \cdot \nabla) V_b + \nabla P|_{y=-1} &= 0. \end{aligned}$$

In a similar way leading to (9.1), we deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|(V, B)\|_{H^{s-\frac{1}{2}}}^2 + \|V_b\|_{H^s}^2) \\ & \leq C \|(\nabla V, \nabla B)\|_{L^\infty} \|(V, B)\|_{H^{s-\frac{1}{2}}}^2 + \|a - 1\|_{H^{s-\frac{1}{2}}} \|B\|_{H^{s-\frac{1}{2}}} \\ & \quad + \|a\zeta\|_{H^{s-\frac{1}{2}}} \|V\|_{H^{s-\frac{1}{2}}} + C \|\nabla V_b\|_{L^\infty} \|V_b\|_{H^s}^2 + \|\nabla P|_{y=-1}\|_{H^s} \|V_b\|_{H^s}, \end{aligned}$$

which together with Lemma 7.9 and Lemma 7.7 yields

$$\begin{aligned} & \frac{d}{dt} (\|(V, B)\|_{H^{s-\frac{1}{2}}}^2 + \|V_b\|_{H^s}^2) \\ & \leq K_\eta A(t)^2 (\|(V, B, V_b)\|_{H^s}^2 + \|\nabla_{x,z} \tilde{v}\|_{H^{s-\frac{1}{2}}(\mathcal{S})}^2 + \|\eta\|_{H^{s+\frac{1}{2}}}^2), \end{aligned}$$

from which and (9.1), we deduce the proposition. \square

Next we present the high order energy estimate.

Proposition 9.4. *Let U be a solution of (6.13). Then there holds*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|D_t U\|_{H^{s-\frac{1}{2}}}^2 + \|T_{\sqrt{a\lambda}} U\|_{H^{s-\frac{1}{2}}}^2) \\ & \leq \langle (f + f_\omega)_{s-1/2}, (D_t U)_{s-1/2} \rangle + K_\eta \mathcal{B}(t) A(t)^2 (\|D_t U\|_{H^{s-\frac{1}{2}}}^2 + \|U\|_{H^s}^2). \end{aligned}$$

Here we denote $f_s \triangleq \langle D \rangle^s f$.

Proof. It follows from (6.13) that $(D_t U)_{s-1/2}$ satisfies

$$\begin{aligned} D_t(D_t U)_{s-1/2} + T_{a\lambda} U_{s-1/2} &= [D_t, \langle D \rangle^{s-1/2}] D_t U + [T_{a\lambda}, \langle D \rangle^{s-1/2}] U \\ & \quad + (f + f_\omega)_{s-1/2} \triangleq F. \end{aligned}$$

Taking L^2 inner product with $(D_t U)_{s-1/2}$, we get

$$\langle D_t(D_t U)_{s-1/2}, (D_t U)_{s-1/2} \rangle + \langle T_{a\lambda} U_{s-1/2}, (D_t U)_{s-1/2} \rangle = \langle F, (D_t U)_{s-1/2} \rangle. \quad (9.2)$$

By integration by parts, we get

$$\begin{aligned} & \langle D_t(D_t U)_{s-1/2}, (D_t U)_{s-1/2} \rangle \\ & = \frac{1}{2} \frac{d}{dt} \langle (D_t U)_{s-1/2}, (D_t U)_{s-1/2} \rangle - \frac{1}{2} \langle \nabla \cdot V (D_t U)_{s-1/2}, (D_t U)_{s-1/2} \rangle \\ & \quad + \langle (T_V \cdot \nabla - V \cdot \nabla) (D_t U)_{s-1/2}, (D_t U)_{s-1/2} \rangle \\ & \geq \frac{1}{2} \frac{d}{dt} \langle (D_t U)_{s-1/2}, (D_t U)_{s-1/2} \rangle - C \|V\|_{B_{\infty,1}^1} \|D_t U\|_{H^{s-\frac{1}{2}}}^2, \end{aligned} \quad (9.3)$$

where we used the inequality

$$\|(T_V \cdot \nabla - V \cdot \nabla) (D_t U)_{s-1/2}\|_{L^2} \leq C \|V\|_{B_{\infty,1}^1} \|D_t U\|_{H^{s-\frac{1}{2}}}.$$

Similarly, we have

$$\begin{aligned}
& \langle T_{a\lambda} U_{s-1/2}, (D_t U)_{s-1/2} \rangle \\
&= \langle (T_{\sqrt{a\lambda}})^* T_{\sqrt{a\lambda}} U_{s-1/2}, (D_t U)_{s-1/2} \rangle + \langle (T_{a\lambda} - (T_{\sqrt{a\lambda}})^* T_{\sqrt{a\lambda}}) U_{s-1/2}, (D_t U)_{s-1/2} \rangle \\
&= \langle (T_{\sqrt{a\lambda}} U_{s-1/2}, T_{\sqrt{a\lambda}} (D_t U)_{s-1/2} \rangle + \langle (T_{a\lambda} - (T_{\sqrt{a\lambda}})^* T_{\sqrt{a\lambda}}) U_{s-1/2}, (D_t U)_{s-1/2} \rangle \\
&= \frac{1}{2} \frac{d}{dt} \langle T_{\sqrt{a\lambda}} U_{s-1/2}, T_{\sqrt{a\lambda}} U_{s-1/2} \rangle - \frac{1}{2} \langle \nabla \cdot V T_{\sqrt{a\lambda}} U_{s-1/2}, T_{\sqrt{a\lambda}} U_{s-1/2} \rangle \\
&\quad + \langle T_{\sqrt{a\lambda}} U_{s-1/2}, [T_{\sqrt{a\lambda}} \langle D \rangle^{s-\frac{1}{2}}, D_t] U \rangle + \langle T_{\sqrt{a\lambda}} U_{s-1/2}, (T_V \cdot \nabla - V \cdot \nabla) T_{\sqrt{a\lambda}} U_{s-1/2} \rangle \\
&\quad + \langle (T_{a\lambda} - (T_{\sqrt{a\lambda}})^* T_{\sqrt{a\lambda}}) U_{s-1/2}, (D_t U)_{s-1/2} \rangle.
\end{aligned}$$

It follows from Proposition 2.6 that

$$\begin{aligned}
& \langle \nabla \cdot V T_{\sqrt{a\lambda}} U_{s-1/2}, T_{\sqrt{a\lambda}} U_{s-1/2} \rangle \leq C \|\nabla V\|_{L^\infty} M_0^{\frac{1}{2}} (\sqrt{a\lambda})^2 \|U\|_{H^s}^2, \\
& \langle (T_{a\lambda} - (T_{\sqrt{a\lambda}})^* T_{\sqrt{a\lambda}}) U_{s-1/2}, (D_t U)_{s-1/2} \rangle \leq C M_0^{\frac{1}{2}} (\sqrt{a\lambda})^2 \|U\|_{H^s} \|D_t U\|_{H^{s-\frac{1}{2}}}, \\
& \langle T_{\sqrt{a\lambda}} U_{s-1/2}, (T_V \cdot \nabla - V \cdot \nabla) T_{\sqrt{a\lambda}} U_{s-1/2} \rangle \leq M_0^{\frac{1}{2}} (\sqrt{a\lambda})^2 \|V\|_{B_{\infty,\infty}^1} \|D_t U\|_{H^{s-\frac{1}{2}}}^2.
\end{aligned}$$

We get by Proposition 2.21 that

$$\begin{aligned}
& \langle T_{\sqrt{a\lambda}} U_{s-1/2}, [T_{\sqrt{a\lambda}} \langle D \rangle^{s-\frac{1}{2}}, D_t] U \rangle \\
& \leq C M_0^{\frac{1}{2}} (\sqrt{a\lambda}) (M_0^{\frac{1}{2}} (\sqrt{a\lambda}) \|V\|_{B_{\infty,\infty}^1} + M_0^{\frac{1}{2}} (D_t \sqrt{a\lambda})) \|U\|_{H^s}^2.
\end{aligned}$$

This proves that

$$\begin{aligned}
& \langle T_{a\lambda} U_{s-1/2}, (D_t U)_{s-1/2} \rangle \geq \frac{1}{2} \frac{d}{dt} \langle T_{\sqrt{a\lambda}} U_{s-1/2}, T_{\sqrt{a\lambda}} U_{s-1/2} \rangle \\
& \quad - C \|V\|_{B_{\infty,\infty}^1} M_0^{\frac{1}{2}} (\sqrt{a\lambda})^2 (\|U\|_{H^s}^2 + \|D_t U\|_{H^{s-\frac{1}{2}}}^2) - M_0^{\frac{1}{2}} (D_t \sqrt{a\lambda}) \|U\|_{H^s}^2 \\
& \quad - C M_0^{\frac{1}{2}} (\sqrt{a\lambda})^2 \|U\|_{H^s} \|D_t U\|_{H^{s-\frac{1}{2}}}. \tag{9.4}
\end{aligned}$$

By Lemma 2.19 and Proposition 2.6, we have

$$\| [D_t, \langle D \rangle^{s-1/2}] D_t U \|_{L^2} \leq C \|V\|_{W^{1,\infty}} \|D_t U\|_{H^{s-\frac{1}{2}}}, \tag{9.5}$$

$$\| [T_{a\lambda}, \langle D \rangle^{s-1/2}] U \|_{L^2} \leq C M_{\frac{1}{2}}^1(a\lambda) \|U\|_{H^s}. \tag{9.6}$$

Then it follows from (9.2)-(9.6) that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|D_t U\|_{H^{s-\frac{1}{2}}}^2 + \|T_{\sqrt{a\lambda}} U\|_{H^{s-\frac{1}{2}}}^2) \leq \langle (f + f_\omega)_{s-1/2}, (D_t U)_{s-1/2} \rangle \\
& \quad + C(1 + \|V\|_{B_{\infty,\infty}^1}) (1 + M_0^{\frac{1}{2}} (\sqrt{a\lambda})^2 + M_{\frac{1}{2}}^1(a\lambda)) (\|D_t U\|_{H^{s-\frac{1}{2}}}^2 + \|U\|_{H^s}^2) \\
& \quad + C M_0^{\frac{1}{2}} (D_t \sqrt{a\lambda}) \|U\|_{H^s}^2,
\end{aligned}$$

while, by Lemma 7.8 and Lemma 4.6, we have

$$\begin{aligned}
& M_0^{\frac{1}{2}} (\sqrt{a\lambda})^2 + M_{\frac{1}{2}}^1(a\lambda) \leq K_\eta A(t)^2, \\
& M_0^{\frac{1}{2}} (D_t \sqrt{a\lambda}) \leq K_\eta \mathcal{B}(t) A(t)^2.
\end{aligned}$$

Then the proposition follows easily. \square

9.3. Energy estimate of the vorticity. Using the equation (6.4), it is easy to see that the vorticity $\tilde{\omega}(t, x, z)$ satisfies

$$\partial_t \tilde{\omega} + \bar{v} \cdot \nabla_{x,z} \tilde{\omega} = \tilde{\omega}^h \cdot \left(\nabla \tilde{v} - \frac{\nabla \rho_\delta}{\partial_z \rho_\delta} \partial_z \tilde{v} \right) + \tilde{\omega}^{d+1} \frac{\partial_z \tilde{v}}{\partial_z \rho_\delta} \triangleq F,$$

where $\tilde{v}^h = (\tilde{v}^1, \dots, \tilde{v}^d)$, $\tilde{\omega}^h = (\tilde{\omega}^1, \dots, \tilde{\omega}^d)$ and

$$\bar{v} = (\tilde{v}^h, \frac{1}{\partial_z \rho_\delta} (\tilde{v}^{d+1} - \partial_t \rho_\delta - \tilde{v}^h \cdot \nabla \rho_\delta)).$$

Proposition 9.5. *Let $s > \frac{d}{2} + 1$ and $s - \frac{1}{2}$ be an integer. Then we have*

$$\frac{d}{dt} \|\tilde{\omega}(t)\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})}^2 \leq K_\eta \mathcal{B}(t) A(t) (\|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})}^2 + \|\eta\|_{H^s}^2 + \|\tilde{v}\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})}^2).$$

We need the following lemma.

Lemma 9.6. *Let $s > 1 + \frac{d}{2}$. Then we have*

$$\begin{aligned} \|\nabla_{x,z} \bar{v}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})} &\leq K_\eta (\|\tilde{v}\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})} + \mathcal{B}(t) \|\eta\|_{H^s}), \\ \|\nabla_{x,z} \bar{v}\|_{L^\infty(\bar{\mathcal{S}})} &\leq K_\eta \mathcal{B}(t). \end{aligned}$$

Proof. Thanks to the definition of \bar{v} , it suffices to consider \bar{v}^{d+1} . Let $\phi = \tilde{v}^{d+1} - \partial_t \rho_\delta - \tilde{v}^h \cdot \nabla \rho_\delta$. Let $\Delta_{x,z}^\delta = \Delta + \delta^{-2} \partial_z^2$ and $\nabla_{x,z}^\delta = (\nabla, \delta^{-1} \partial_z)$. Using the fact that

$$\Delta_{x,z}^\delta \rho_\delta = 2\delta^{-1} e^{\delta z|D|} |D| \eta \triangleq f_\eta,$$

we find that

$$\Delta_{x,z}^\delta \phi = \Delta_{x,z}^\delta \tilde{v}^{d+1} - \partial_t f_\eta - \Delta_{x,z}^\delta \tilde{v}^h \cdot \nabla \rho_\delta - 2\nabla_{x,z}^\delta \tilde{v}^h \cdot \nabla \nabla_{x,z}^\delta \rho_\delta - \tilde{v} \cdot \nabla f_\eta \triangleq F_{\eta,v}$$

together with the boundary condition $\phi = 0$ on $z = 0$ and $z = -1$.

By (1.4), we have

$$\|\partial_t f_\eta\|_{H^{s-\frac{3}{2}}(\bar{\mathcal{S}})} \leq C \|\partial_t \eta\|_{H^{s-1}} \leq C (\|\nabla \eta\|_{L^\infty} \|V\|_{H^{s-1}} + \|V\|_{L^\infty} \|\eta\|_{H^s} + \|B\|_{H^{s-1}}),$$

which together with Lemma 2.13, Lemma 4.3 implies that

$$\|F_{\eta,v}\|_{H^{s-\frac{3}{2}}(\bar{\mathcal{S}})} \leq K_\eta (\|\tilde{v}\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})} + \|\tilde{v}\|_{W^{1,\infty}(\bar{\mathcal{S}})} \|\eta\|_{H^s}).$$

Then Proposition 4.1 ensures that

$$\|\nabla_{x,z} \phi\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})} \leq K_\eta (\|\tilde{v}\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})} + \|\tilde{v}\|_{W^{1,\infty}(\bar{\mathcal{S}})} \|\eta\|_{H^s}). \quad (9.7)$$

We write

$$\begin{aligned} F_{\eta,v} &= \nabla_{x,z}^\delta \cdot (\nabla_{x,z}^\delta \tilde{v}_{d+1} - \nabla_{x,z}^\delta \tilde{v}^h \cdot \nabla \rho_\delta + (\tilde{v}^h, 0) f_\eta) \\ &\quad + (-\partial_t f_\eta - \nabla_{x,z}^\delta \tilde{v}^h \cdot \nabla_{x,z}^\delta \nabla \rho_\delta + \nabla \cdot \tilde{v}^h f_\eta) \\ &= \nabla_{x,z}^\delta \cdot F_{\eta,v}^1 + F_{\eta,v}^2. \end{aligned}$$

By Lemma 2.17 and Lemma 2.18, we have

$$\begin{aligned} \|T_{\nabla_{x,z}^\delta \tilde{v}^h \cdot \nabla \rho_\delta}\|_{L_z^\infty(I; B_{\infty,1}^0)} &\leq C \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})} \|\nabla \rho_\delta\|_{L_z^\infty(I; B_{\infty,1}^0)}, \\ \|T_{\nabla \rho_\delta \nabla_{x,z}^\delta \tilde{v}^h}\|_{L_z^\infty(I; B_{\infty,1}^0)} &\leq C \|\nabla \rho_\delta\|_{L^\infty(\bar{\mathcal{S}})} \|\nabla_{x,z} \tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^0)}, \\ \|R(\nabla_{x,z}^\delta \tilde{v}^h, \nabla \rho_\delta)\|_{L_z^\infty(I; B_{\infty,1}^0)} &\leq C \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})} \|\nabla \rho_\delta\|_{L_z^\infty(I; B_{\infty,1}^\epsilon)}, \end{aligned}$$

for any $\epsilon > 0$, which gives

$$\|\nabla_{x,z}^\delta \tilde{v}^h \cdot \rho_\delta\|_{L_z^\infty(I; B_{\infty,1}^0)} \leq K_\eta \|\tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^1)}.$$

The same estimate holds for $(\tilde{v}^h, 0)f_\eta$. Hence,

$$\|F_{\eta,v}^1\|_{L_z^\infty(I; B_{\infty,1}^0)} \leq K_\eta \|\tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^1)}.$$

By Lemma 2.17 and Lemma 2.18 again, we have

$$\begin{aligned} \|T_{\nabla_{x,z}^\delta \tilde{v}^h} \nabla_{x,z}^\delta \nabla \rho_\delta\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})} &\leq C \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})} \|\nabla_{x,z}^\delta \rho_\delta\|_{\tilde{L}_z^2(I; B_{\infty,1}^{\frac{1}{2}})}, \\ \|T_{\nabla_{x,z}^\delta \nabla \rho_\delta} \nabla_{x,z}^\delta \tilde{v}^h\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})} &\leq C \|\nabla_{x,z} \tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^0)} \|\nabla_{x,z}^\delta \nabla \rho_\delta\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})}, \\ \|R(\nabla_{x,z}^\delta \nabla \rho_\delta, \nabla_{x,z}^\delta \tilde{v}^h)\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})} &\leq C \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})} \|\nabla_{x,z}^\delta \rho_\delta\|_{\tilde{L}_z^2(I; C^{1+\epsilon})}, \end{aligned}$$

for any $\epsilon > 0$, which gives

$$\|\nabla_{x,z}^\delta \tilde{v}^h \nabla_{x,z}^\delta \nabla \rho_\delta\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})} \leq K_\eta \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})}.$$

The same estimate holds for $\nabla \cdot \tilde{v}^h f_\eta$ and $\partial_t f_\eta$. Hence,

$$\|F_{\eta,v}^2\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})} \leq K_\eta \|\tilde{v}\|_{W^{1,\infty}(\bar{\mathcal{S}})}.$$

Then we apply Proposition 4.2 to conclude that

$$\|\phi\|_{L_z^\infty(I; B_{\infty,1}^1)} \leq K_\eta \|\tilde{v}\|_{L_z^\infty(I; B_{\infty,1}^1)}. \quad (9.8)$$

Next we turn to the estimate of \bar{v}^{d+1} , which satisfies

$$\Delta_{x,z}^\delta \bar{v}^{d+1} = \left(\frac{2|\nabla_{x,z}^\delta \partial_z \rho_\delta|^2}{(\partial_z \rho_\delta)^3} - \frac{\partial_z f_\eta}{(\partial_z \rho_\delta)^2} \right) \phi + 2\nabla_{x,z}^\delta \left(\frac{1}{\partial_z \rho_\delta} \right) \cdot \nabla_{x,z} \phi + \frac{1}{\partial_z \rho_\delta} \Delta_{x,z}^\delta \phi \triangleq G_{\eta,v}$$

with $\bar{v}_{d+1}|_{z=0} = 0$ and $\bar{v}_{d+1}|_{z=-1} = 0$. By Lemma 2.13 and Lemma 4.3, we have

$$\|G_{\eta,v}\|_{H^{s-\frac{3}{2}}(\bar{\mathcal{S}})} \leq K_\eta (\|\phi\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})} + \|\nabla_{x,z} \phi\|_{L^\infty(\bar{\mathcal{S}})} \|\eta\|_{H^s}),$$

from which, (9.7)-(9.8) and Proposition 4.1, we deduce the first inequality.

We write

$$\begin{aligned} G_{\eta,v} &= \left(\frac{2|\nabla_{x,z}^\delta \partial_z \rho_\delta|^2}{(\partial_z \rho_\delta)^3} - \frac{\partial_z f_\eta}{(\partial_z \rho_\delta)^2} \right) \phi + \nabla_{x,z}^\delta \left(\frac{1}{\partial_z \rho_\delta} \right) \cdot \nabla_{x,z} \phi + \nabla_{x,z}^\delta \cdot \left(\frac{1}{\partial_z \rho_\delta} \nabla_{x,z}^\delta \phi \right) \\ &\triangleq G_{\eta,v}^1 + \nabla_{x,z}^\delta \cdot G_{\eta,v}^2. \end{aligned}$$

Then a similar argument leading to (9.8) yields

$$\|G_{\eta,v}^1\|_{\tilde{L}_z^2(I; B_{\infty,1}^{-\frac{1}{2}})} + \|G_{\eta,v}^2\|_{L_z^\infty(I; B_{\infty,1}^0)} \leq K_\eta \|\nabla_{x,z} \phi\|_{L_z^\infty(I; B_{\infty,1}^0)}.$$

Then Proposition 4.2 together with (9.8) gives the second inequality. \square

Now we are in position to prove Proposition 9.5.

Proof of Proposition 9.5: Let $k \in [0, s - \frac{1}{2}]$ be an interger. Then we have

$$\frac{d}{dt} \|\nabla_{x,z}^k \tilde{\omega}\|_{L^2(\bar{\mathcal{S}})}^2 = 2 \langle -\nabla_{x,z}^k (\bar{v} \cdot \nabla_{x,z} \tilde{\omega}) + \nabla_{x,z}^k F, \nabla_{x,z}^k \tilde{\omega} \rangle.$$

First of all, we have by Lemma 2.13 that

$$\begin{aligned} \|\nabla_{x,z}^k F\|_{L^2(\bar{\mathcal{S}})} &\leq K_\eta \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})} (\|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})}) \\ &\quad + K_\eta \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})}^2 \|\eta\|_{H^s}. \end{aligned}$$

Thanks to $\bar{v}^3 = 0$ on $z = 0$ and $z = -1$, we have

$$\langle \nabla_{x,z}^k (\bar{v} \cdot \nabla_{x,z} \tilde{\omega}), \nabla_{x,z}^k \tilde{\omega} \rangle = \langle [\nabla_{x,z}^k \bar{v}] \cdot \nabla_{x,z} \tilde{\omega}, \nabla_{x,z}^k \tilde{\omega} \rangle - \langle \operatorname{div}_{x,z} \bar{v} \nabla_{x,z}^k \tilde{\omega}, \nabla_{x,z}^k \tilde{\omega} \rangle.$$

Then we deduce from Lemma 2.20 and Lemma 9.6 that

$$\begin{aligned} \langle \nabla_{x,z}^k (\bar{v} \cdot \nabla_{x,z} \tilde{\omega}), \nabla_{x,z}^k \tilde{\omega} \rangle &\leq C \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\bar{\mathcal{S}})} \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})} \|\nabla_{x,z} \bar{v}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})} \\ &\quad + C \|\nabla_{x,z} \bar{v}\|_{L^\infty(\bar{\mathcal{S}})} \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})}^2 \\ &\leq K_\eta A(t) \mathcal{B}(t) (\|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})}^2 + \|\eta\|_{H^s}^2 + \|\tilde{v}\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})}^2). \end{aligned}$$

Summing up, we conclude the proposition. \square

9.4. Nonlinear estimates. Recall that the nonlinear term f is given by

$$f = D_t h_1 + (T_{a\lambda} - T_a T_\lambda) U + [T_a, D_t] \zeta + D_t [D_t, T_\zeta] B - T_a h_2,$$

where

$$\begin{aligned} h_1 &= (T_V - V) \cdot \nabla V - R(a, \zeta) + T_\zeta (T_V - V) \cdot \nabla B, \\ h_2 &= (T_V - V) \cdot \nabla \zeta + [T_\zeta, T_\lambda] B + (\zeta - T_\zeta) T_\lambda B + R(\eta) V + \zeta R(\eta) B. \end{aligned}$$

Lemma 9.7. *It holds that*

$$\|h_1\|_{H^{s-\frac{1}{2}}} \leq K_\eta A(t)^2 (\|V\|_{H^{s-\frac{1}{2}}} + \|\eta\|_{H^s}).$$

Proof. It follows from Lemma 2.10 that

$$\|h_1\|_{H^{s-\frac{1}{2}}} \leq C \|\nabla V\|_{L^\infty} \|V\|_{H^{s-\frac{1}{2}}} + C \|a\|_{C^{\frac{1}{2}}} \|\zeta\|_{H^{s-1}} + K_\eta \|\nabla B\|_{L^\infty} \|V\|_{H^{s-\frac{1}{2}}},$$

which along with Lemma 7.8 gives the lemma. \square

Lemma 9.8. *It holds that*

$$\|D_t h_1\|_{H^{s-\frac{1}{2}}} \leq K_\eta \mathcal{B}(t) A(t)^2 (\|(V, B)\|_{H^s} + \|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-\frac{1}{2}})} + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

Proof. First, we consider the second term of the h_1 . We denote $\bar{\partial}_t = \partial_t + V \cdot \nabla$. By Lemma 2.15, we get

$$\begin{aligned} \|\bar{\partial}_t R(a-1, \zeta)\|_{H^{s-\frac{1}{2}}} &\leq C (\|\bar{\partial}_t a\|_{L^\infty} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\bar{\partial}_t \zeta\|_{L^\infty} \|a-1\|_{H^{s-\frac{1}{2}}}) \\ &\quad + \|\nabla V\|_{L^\infty} (\|a\|_{L^\infty} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\zeta\|_{L^\infty} \|a-1\|_{H^{s-\frac{1}{2}}}) + \|V\|_{H^s} \|a\|_{L^\infty} \|\zeta\|_{C^{\frac{1}{2}+\varepsilon}}. \end{aligned}$$

Using the fact that

$$\bar{\partial}_t a = D_t a + (V - T_V) \cdot \nabla a,$$

we get by Lemma 2.10 and Lemma 7.8 that

$$\|\bar{\partial}_t a\|_{L^\infty} \leq \|D_t a\|_{L^\infty} + \|V\|_{W^{1,\infty}} \|a\|_{L^\infty} \leq K_\eta \mathcal{B}(t) A(t)^2.$$

Note that $\bar{\partial}_t \zeta = \nabla B - \nabla V \cdot \nabla \eta$, hence,

$$\|\bar{\partial}_t \zeta\|_{L^\infty} \leq K_\eta A(t).$$

Then by Lemma 7.8, we obtain

$$\|\bar{\partial}_t R(a-1, \zeta)\|_{H^{s-\frac{1}{2}}} \leq K_\eta \mathcal{B}(t) A(t)^2 (\|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-\frac{1}{2}})} + \|V\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

which implies that

$$\begin{aligned} \|D_t R(a-1, \zeta)\|_{H^{s-\frac{1}{2}}} &\leq \|\bar{\partial}_t R(a-1, \zeta)\|_{H^{s-\frac{1}{2}}} + C \|R(a-1, \zeta)\|_{C^{\frac{1}{2}}} \|V\|_{H^s} \\ &\leq K_\eta \mathcal{B}(t) A(t)^2 (\|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-\frac{1}{2}})} + \|V\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

First, we consider the first and third terms of the h_1 . We write

$$D_t(T_V - V) \cdot \nabla V = -[D_t, T_{\nabla V}]V - T_{\nabla V} D_t V - D_t R(\nabla V, V).$$

By Remark 2.22, we have

$$\|[D_t, T_{\nabla V}]V\|_{H^{s-\frac{1}{2}}} \leq C \|V\|_{B_{\infty,1}^1} \|V\|_{H^{s-\frac{1}{2}}} + C \|D_t \nabla V\|_{C^{-\frac{1}{2}}} \|V\|_{H^s}.$$

While, using the equation

$$D_t \nabla V = \nabla D_t V - T_{\nabla V} \cdot \nabla V = \nabla(a\zeta) + \nabla(T_V - V) \cdot \nabla V - T_{\nabla V} \cdot \nabla V,$$

we deduce from Lemma 2.10 and Lemma 7.8 that

$$\|D_t \nabla V\|_{C^{-\frac{1}{2}}} \leq K_\eta \|a\|_{C^{\frac{1}{2}}} + C \|V\|_{W^{1,\infty}}^2 \leq K_\eta A(t)^2.$$

This shows that

$$\|[D_t, T_{\nabla V}]V\|_{H^{s-\frac{1}{2}}} \leq C \|V\|_{B_{\infty,1}^1} \|V\|_{H^{s-\frac{1}{2}}} + K_\eta A(t)^2 \|V\|_{H^s}.$$

We get by Lemma 2.10 that

$$\|T_{\nabla V} D_t V\|_{H^{s-\frac{1}{2}}} \leq C \|V\|_{W^{1,\infty}} \|D_t V\|_{H^{s-\frac{1}{2}}}.$$

It follows from Lemma 2.15 with $u = V$ and $v = \nabla V$ that

$$\begin{aligned} \|\bar{\partial}_t R(\nabla V, V)\|_{H^{s-\frac{1}{2}}} &\leq C (\|\bar{\partial}_t \nabla V\|_{C^{-\frac{1}{2}}} + \|\bar{\partial}_t V\|_{C^{\frac{1}{2}}} + \|V\|_{W^{1,\infty}}^2) \|V\|_{H^s} \\ &\leq K_\eta A(t)^2 \|V\|_{H^s}, \end{aligned}$$

which ensures that

$$\begin{aligned} \|D_t R(\nabla V, V)\|_{H^{s-\frac{1}{2}}} &\leq \|\bar{\partial}_t R(\nabla V, V)\|_{H^{s-\frac{1}{2}}} + \|R(\nabla V, V)\|_{C^{\frac{1}{2}}} \|V\|_{H^s} \\ &\leq K_\eta A(t)^2 \|V\|_{H^s}. \end{aligned}$$

Thus, we obtain

$$\|D_t(T_V - V) \cdot \nabla V\|_{H^{s-\frac{1}{2}}} \leq K_\eta \mathcal{B}(t) A(t)^2 \|V\|_{H^s}.$$

We write

$$D_t T_\zeta(T_V - V) \cdot \nabla B = [D_t, T_\zeta](T_V - V) \cdot \nabla B + T_\zeta D_t(T_V - V) \cdot \nabla B$$

In a similar way as the above, we can deduce that

$$\|D_t T_\zeta(T_V - V) \cdot \nabla B\|_{H^{s-\frac{1}{2}}} \leq K_\eta \mathcal{B}(t) A(t)^2 \|(V, B)\|_{H^s}.$$

Putting the above estimates together gives the lemma. \square

Lemma 9.9. *It holds that*

$$\|h_2\|_{H^{s-\frac{1}{2}}} \leq K_\eta \mathcal{B}(t) A(t) (\|(V, B)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

Proof. By Lemma 2.10, we get

$$\|(T_V - V) \cdot \nabla \zeta\|_{H^{s-\frac{1}{2}}} \leq C \|\zeta\|_{C^{\frac{1}{2}}} \|V\|_{H^s} \leq K_\eta \|V\|_{H^s}.$$

By Proposition 2.6 and Lemma 4.6, we have

$$\|[T_\zeta, T_\lambda]B\|_{H^{s-\frac{1}{2}}} \leq K_\eta \|B\|_{H^s}.$$

It follows from Proposition 8.3 that

$$\|R(\eta)V\|_{H^{s-\frac{1}{2}}} \leq K_\eta (\|V\|_{H^s} + (\|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)} + A(t)) \|\eta\|_{H^{s+\frac{1}{2}}}).$$

By Lemma 2.11 and Proposition 8.3, we get

$$\begin{aligned} \|\zeta R(\eta)B\|_{H^{s-\frac{1}{2}}} &\leq C \|R(\eta)B\|_{L^\infty} \|\eta\|_{H^{s+\frac{1}{2}}} + C \|\zeta\|_{L^\infty} \|R(\eta)B\|_{H^{s-\frac{1}{2}}} \\ &\leq K_\eta (1 + \|\tilde{v}\|_{L^\infty(I; B_{\infty,1}^1)} + A(t)) (\|B\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

By Lemma 2.10 and Proposition 2.7, we have

$$\|(\zeta - T_\zeta)T_\lambda B\|_{H^{s-\frac{1}{2}}} \leq C \|T_\lambda B\|_{L^\infty} \|\eta\|_{H^{s+\frac{1}{2}}} \leq K_\eta \|B\|_{B_{\infty,1}^1} \|\eta\|_{H^{s+\frac{1}{2}}}.$$

This completes the proof of the lemma. \square

Lemma 9.10. *Let $f_1 = f - D_t[D_t, T_\zeta]B$. It holds that*

$$\|f_1\|_{H^{s-\frac{1}{2}}} \leq K_\eta \mathcal{B}(t) A(t)^3 (\|(V, B)\|_{H^s} + \|\nabla_{x,z} \tilde{v}\|_{L_x^2(I; H^{s-\frac{1}{2}}}) + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

Proof. By Proposition 2.6, Lemma 7.8 and Lemma 4.6, we get

$$\begin{aligned} \|(T_{a\lambda} - T_a T_\lambda)U\|_{H^{s-\frac{1}{2}}} &\leq C M_{\frac{1}{2}}^1(\lambda) M_{\frac{1}{2}}^0(a) \|U\|_{H^s} \\ &\leq K_\eta A(t)^2 \|U\|_{H^s} \leq K_\eta A(t)^2 \|(V, B)\|_{H^s}. \end{aligned}$$

It follows from Proposition 2.21 and Lemma 7.8 that

$$\begin{aligned} \|[T_a, D_t]\zeta\|_{H^{s-\frac{1}{2}}} &\leq C (M_0^0(a) \|V\|_{B_{\infty,1}^1} + M_0^0(D_t a)) \|\eta\|_{H^{s+\frac{1}{2}}} \\ &\leq K_\eta \mathcal{B}(t) A(t)^2 \|\eta\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

By Lemma 2.10, Lemma 7.8 and Lemma 9.9, we get

$$\begin{aligned} \|T_a h_2\|_{H^{s-\frac{1}{2}}} &\leq C \|a\|_{L^\infty} \|h_2\|_{H^{s-\frac{1}{2}}} \\ &\leq K_\eta \mathcal{B}(t) A(t)^3 (\|(V, B)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}}). \end{aligned}$$

Finally, the estimate for $D_t h_1$ follows from Lemma 9.9. \square

Next we estimate $f_\omega = -T_a R_\omega$, where R_ω is defined by

$$\begin{aligned} R_\omega^i &= (\partial_y v_\omega^i - \partial_{x_j} v_\omega^i \cdot \partial_{x_j} \eta) + \partial_{x_i} \eta (\partial_y v_\omega^{d+1} - \partial_{x_j} \eta \partial_{x_j} v_\omega^{d+1}) \\ &\quad + (\omega_{i,d+1} - \partial_{x_j} \eta \omega_{ij} + \partial_{x_i} \eta \partial_{x_j} \eta \omega_{j,d+1})|_{y=\eta}. \end{aligned}$$

Lemma 9.11. *It holds that*

$$\|f_\omega\|_{H^{s-1}} \leq K_\eta A(t)^3 (\|V_b\|_{H^{\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} + \|\eta\|_{H^s}).$$

Proof. Let $I_0 = [-\frac{1}{2}, 0]$. By Lemma 2.11 and Lemma 2.10, it is easy to see that

$$\begin{aligned} \|R_\omega\|_{H^{s-1}} &\leq K_\eta (\|\nabla_{x,z} \tilde{v}_\omega\|_{L^\infty(I_0; H^{s-1})} + \|\tilde{\omega}\|_{L^\infty(I_0; H^{s-1})}) \\ &\quad + C(\|\nabla_{x,z} \tilde{v}_\omega\|_{L_z^\infty(I_0; C^0)} + \|\omega\|_{L^\infty(\Omega_t)}) \|\eta\|_{H^s}, \end{aligned}$$

from which and Lemma 8.4, we deduce that

$$\|R_\omega\|_{H^{s-1}} \leq K_\eta A(t) (\|V_b\|_{H^{\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} + \|\eta\|_{H^s}). \quad (9.9)$$

Then the lemma follows from Lemma 2.10 and Lemma 7.8. \square

Lemma 9.12. *It holds that*

$$\|D_t f_\omega\|_{H^{s-1}} \leq K_\eta \mathcal{B}(t) A(t)^4 (\|(V, B, V_b)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}).$$

Proof. Let $\bar{\partial}_t = \partial_t + V \cdot \nabla$. Then $\bar{\partial}_t R_\omega$ could be written as

$$\begin{aligned} \bar{\partial}_t R_\omega &= g_1(\nabla \eta)(\partial_t + v \cdot \nabla_{x,y}) \nabla_{x,y} v_\omega + g_2(\nabla \eta) \bar{\partial}_t \nabla \eta \nabla_{x,z} v_\omega \\ &\quad + g_3(\nabla \eta)(\partial_t + v \cdot \nabla_{x,y}) \omega + g_4(\nabla \eta) \bar{\partial}_t \nabla \eta \omega|_{y=\eta}, \end{aligned}$$

for some smooth functions $g_i (i = 1, 2, 3, 4)$. Then it follows from Lemma 2.11 that

$$\begin{aligned} \|\bar{\partial}_t R_\omega\|_{H^{s-1}} &\leq K_\eta (\|(\partial_t + v \cdot \nabla_{x,y}) \nabla_{x,y} v_\omega|_{y=\eta}\|_{H^{s-1}} + \|(\partial_t + v \cdot \nabla_{x,y}) \omega|_{y=\eta}\|_{H^{s-1}}) \\ &\quad + K_\eta (\|(\partial_t + v \cdot \nabla_{x,y}) \nabla_{x,y} v_\omega|_{y=\eta}\|_{C^0} + \|(\partial_t + v \cdot \nabla_{x,y}) \omega\|_{L^\infty(\Omega_t)}) \|\eta\|_{H^{s+\frac{1}{2}}} \\ &\quad + K_\eta \|\bar{\partial}_t \nabla \eta\|_{L^\infty} (\|\nabla_{x,y} v_\omega|_{y=\eta}\|_{H^{s-1}} + \|\omega|_{y=\eta}\|_{H^{s-1}}) \\ &\quad + K_\eta (\|\nabla_{x,y} v_\omega|_{y=\eta}\|_{L^\infty} + \|\omega\|_{L^\infty(\Omega_t)}) \|\bar{\partial}_t \nabla \eta\|_{H^{s-1}} \\ &\quad + K_\eta (\|\nabla_{x,y} v_\omega|_{y=\eta}\|_{L^\infty} + \|\omega\|_{L^\infty(\Omega_t)}) \|\bar{\partial}_t \nabla \eta\|_{L^\infty} \|\eta\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

Let $I_0 = [-\frac{1}{2}, 0]$. By Lemma 2.11 again, we get

$$\begin{aligned} \|(\partial_t + v \cdot \nabla_{x,y}) \nabla_{x,y} v_\omega|_{y=\eta}\|_{H^{s-1}} &\leq K_\eta (\|\nabla_{x,z} \tilde{v}_\omega\|_{X^{s-1}(I_0)} + \|\nabla_{x,z} \tilde{v}_\omega\|_{L_z^\infty(I_0; L^\infty)} \|\nabla_{x,z} \tilde{v}\|_{X^{s-1}(I_0)} \\ &\quad + \|\nabla_{x,z} \tilde{v}_\omega\|_{X^{s-1}(I_0)} \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} + \|\nabla_{x,z} \tilde{v}_\omega\|_{L_z^\infty(I_0; L^\infty)} \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\eta\|_{H^s}). \end{aligned}$$

Here $\dot{v}_\omega = (\partial_t + v \cdot \nabla_{x,y}) v_\omega$. Using the equation $(\partial_t + v \cdot \nabla_{x,y}) \omega = \omega \cdot \nabla_{x,y} v$ and Lemma 2.11, we infer that

$$\begin{aligned} \|(\partial_t + v \cdot \nabla_{x,y}) \omega|_{y=\eta}\|_{H^{s-\frac{1}{2}}} &\leq K_\eta (\|\nabla_{x,y} v\|_{L^\infty(\Omega_t)} \|\nabla_{x,z} \tilde{v}\|_{X^{s-1}(I_0)} \\ &\quad + \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2 \|\eta\|_{H^s}). \end{aligned}$$

Using the equation $\bar{\partial}_t \nabla \eta = \nabla B - \nabla V \cdot \nabla \eta$ and Lemma 2.11, we get

$$\|\bar{\partial}_t \nabla \eta\|_{H^{s-1}} \leq K_\eta (\|(V, B)\|_{H^s} + \|\nabla V\|_{L^\infty} \|\eta\|_{H^s}).$$

On the other hand, we have

$$\|(\partial_t + v \cdot \nabla_{x,y}) \nabla_{x,y} v_\omega|_{y=\eta}\|_{C^0}$$

$$\begin{aligned}
&\leq K_\eta (\|\nabla_{x,z} \widetilde{v_\omega}\|_{L_z^\infty(I_0; C^0)} + \|\nabla_{x,z} \widetilde{v_\omega}\|_{L_z^\infty(I_0; L^\infty)} \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}), \\
&\|(\partial_t + v \cdot \nabla_{x,y}) \omega|_{y=\eta}\|_{L^\infty} \leq K_\eta \|\nabla_{x,y} v\|_{L^\infty(\Omega_t)}^2, \\
&\|\bar{\partial}_t \nabla \eta|_{y=\eta}\|_{L^\infty} \leq K_\eta \|(\nabla B, \nabla V)\|_{L^\infty}.
\end{aligned}$$

Summing up the above estimates, we apply Lemma 8.4 and Lemma 8.5 to obtain

$$\|\bar{\partial}_t R_\omega\|_{H^{s-1}} \leq K_\eta \mathcal{B}(t) A(t)^2 (\|(V, B, V_b)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}),$$

which implies that

$$\begin{aligned}
\|D_t R_\omega\|_{H^{s-1}} &\leq \|\bar{\partial}_t R_\omega\|_{H^{s-1}} + C \|R_\omega\|_{L^\infty} \|V\|_{H^s} \\
&\leq K_\eta \mathcal{B}(t) A(t)^2 (\|(V, B, V_b)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}).
\end{aligned}$$

which along with Proposition 2.21, Lemma 7.8 and (9.9) gives

$$\begin{aligned}
\|D_t f_\omega\|_{H^{s-1}} &\leq \| [D_t, T_a] R_\omega \|_{H^{s-1}} + \| T_a D_t R_\omega \|_{H^{s-1}} \\
&\leq K_\eta \mathcal{B}(t) A(t)^4 (\|(V, B, V_b)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}).
\end{aligned}$$

The proof is finished. \square

Lemma 9.13. *It holds that*

$$\begin{aligned}
\langle (f_\omega)_{s-1/2}, (D_t U)_{s-1/2} \rangle &\leq \frac{d}{dt} \langle (f_\omega)_{s-1}, U_s \rangle \\
&\quad + K_\eta \mathcal{B}(t) A(t)^4 (\|(V, B, V_b)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}) \|U\|_{H^s}.
\end{aligned}$$

Proof. A direct calculation yields

$$\begin{aligned}
\langle (f_\omega)_{s-1/2}, (D_t U)_{s-1/2} \rangle &= - \langle (f_\omega)_{s-1/2}, [T_V \cdot \nabla, \langle D \rangle^{s-1/2}] U \rangle + \langle (f_\omega)_{s-1/2}, D_t U_{s-1/2} \rangle \\
&= - \langle (f_\omega)_{s-1/2}, [T_V \cdot \nabla, \langle D \rangle^{s-1/2}] U \rangle + \frac{d}{dt} \langle (f_\omega)_{s-1}, U_s \rangle \\
&\quad - \langle [T_V \cdot \nabla, \langle D \rangle^{s-1/2}] f_\omega, U_{s-1/2} \rangle - \langle (D_t f_\omega)_{s-1}, U_s \rangle \\
&\quad + \langle ((T_V \cdot \nabla)^* + T_V \cdot \nabla)(f_\omega)_{s-1/2}, U_{s-1/2} \rangle.
\end{aligned}$$

By Lemma 2.19 and Proposition 2.6, we have

$$\begin{aligned}
\|[T_V \cdot \nabla, \langle D \rangle^{s-1/2}] U\|_{H^{\frac{1}{2}}} &\leq C \|V\|_{W^{1,\infty}} \|U\|_{H^s}, \\
\|[T_V \cdot \nabla, \langle D \rangle^{s-1/2}] f_\omega\|_{H^{-\frac{1}{2}}} &\leq C \|V\|_{W^{1,\infty}} \|f_\omega\|_{H^{s-1}}, \\
\|((T_V \cdot \nabla)^* + T_V \cdot \nabla)(f_\omega)_{s-1/2}\|_{H^{-\frac{1}{2}}} &\leq C \|V\|_{W^{1,\infty}} \|f_\omega\|_{H^{s-1}}.
\end{aligned}$$

Then the lemma follows from Lemma 9.11 and Lemma 9.12. \square

Lemma 9.14. *It holds that*

$$\begin{aligned}
\langle (D_t [D_t, T_\zeta] B)_{s-\frac{1}{2}}, (D_t U)_{s-1/2} \rangle &= \frac{d}{dt} \left(\langle g_{s-1/2}, h_{s-1/2} \rangle + \frac{1}{2} \langle g_{s-1/2}, g_{s-1/2} \rangle \right) \\
&\quad + K_\eta \mathcal{B}(t) A(t)^3 (\|D_t U\|_{H^{s-\frac{1}{2}}} + \|(V, B)\|_{H^s} + \|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-\frac{1}{2}})} + \|\eta\|_{H^{s+\frac{1}{2}}})^2.
\end{aligned}$$

where $g = [D_t, T_\zeta] B$ and $h = D_t U - [D_t, T_\zeta] B$.

Proof. A direct calculation yields

$$\begin{aligned}
\langle (D_t g)_{s-1/2}, (D_t U)_{s-1/2} \rangle &= \langle (D_t g)_{s-1/2}, h_{s-1/2} \rangle + \langle (D_t g)_{s-1/2}, g_{s-1/2} \rangle \\
&= \frac{d}{dt} \left(\langle g_{s-1/2}, h_{s-1/2} \rangle + \frac{1}{2} \langle g_{s-1/2}, g_{s-1/2} \rangle \right)
\end{aligned}$$

$$\begin{aligned}
& - \langle [T_V \cdot \nabla, \langle D \rangle^{s-1/2}] g, h_{s-1/2} \rangle + \langle g_{s-1/2}, (D_t)^* h_{s-1/2} \rangle \\
& - \langle [T_V \cdot \nabla, \langle D \rangle^{s-1/2}] g, g_{s-1/2} \rangle + \frac{1}{2} \langle (T_V \cdot \nabla)^* + T_V \cdot \nabla g_{s-1/2}, g_{s-1/2} \rangle \\
& \geq \frac{d}{dt} \left(\langle g_{s-1/2}, h_{s-1/2} \rangle + \frac{1}{2} \langle g_{s-1/2}, g_{s-1/2} \rangle \right) \\
& - \| [T_V \cdot \nabla, \langle D \rangle^{s-1/2}] g \|_{L^2} (\|h\|_{H^{s-\frac{1}{2}}} + \|g\|_{H^{s-\frac{1}{2}}}) \\
& - \| (T_V \cdot \nabla)^* + T_V \cdot \nabla g_{s-1/2} \|_{L^2} \|g\|_{H^{s-\frac{1}{2}}} - \|g\|_{H^s} \|(D_t)^* h_{s-1/2}\|_{H^{-\frac{1}{2}}}.
\end{aligned}$$

It follows from Proposition 2.21 that

$$\begin{aligned}
\|g\|_{H^{s-\frac{1}{2}}} & \leq C(\|V\|_{W^{1,\infty}} + \|D_t \zeta\|_{L^\infty}) \|B\|_{H^{s-\frac{1}{2}}} \leq K_\eta A(t) \|B\|_{H^{s-\frac{1}{2}}}, \\
\|g\|_{H^s} & \leq C(\|V\|_{W^{1,\infty}} + \|D_t \zeta\|_{L^\infty}) \|B\|_{H^s} \leq K_\eta A(t) \|B\|_{H^s},
\end{aligned}$$

from which, Proposition 2.6 and Lemma 2.19, we infer

$$\begin{aligned}
\| [T_V \cdot \nabla, \langle D \rangle^{s-1/2}] g \|_{L^2} & \leq K_\eta A(t)^2 \|B\|_{H^{s-\frac{1}{2}}}, \\
\| ((T_V \cdot \nabla)^* + T_V \cdot \nabla) g_{s-1/2} \|_{L^2} & \leq K_\eta A(t)^2 \|B\|_{H^{s-\frac{1}{2}}}, \\
\|h\|_{H^{s-\frac{1}{2}}} & \leq \|D_t U\|_{H^{s-\frac{1}{2}}} + K_\eta A(t) \|B\|_{H^{s-\frac{1}{2}}}.
\end{aligned}$$

We have by Lemma 2.19 and Lemma 2.10 that

$$\begin{aligned}
\|(D_t)^* h_{s-1/2}\|_{H^{-\frac{1}{2}}} & \leq \|D_t h\|_{H^{s-1}} + \| [T_V \cdot \nabla, \langle D \rangle^{s-\frac{1}{2}}] h \|_{L^2} + \|T_{\nabla \cdot V} h_{s-1/2}\|_{L^2} \\
& \leq \|D_t h\|_{H^{s-1}} + C\|V\|_{W^{1,\infty}} \|h\|_{H^{s-\frac{1}{2}}} \\
& \leq \|D_t h\|_{H^{s-1}} + K_\eta A(t)^2 (\|D_t U\|_{H^{s-\frac{1}{2}}} + \|B\|_{H^{s-\frac{1}{2}}}).
\end{aligned}$$

While, by the equation (6.12), we find

$$D_t h = D_t h_1 - [D_t, T_a] \zeta - T_a D_t \zeta.$$

Then we get by Lemma 9.8, Proposition 2.21 and Lemma 7.8 that

$$\|D_t h\|_{H^{s-1}} \leq K_\eta \mathcal{B}(t) A(t)^2 (\|(V, B)\|_{H^s} + \|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-\frac{1}{2}}}) + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

This shows that

$$\begin{aligned}
\|(D_t)^* h_{s-1/2}\|_{H^{-\frac{1}{2}}} & \leq K_\eta \mathcal{B}(t) A(t)^2 (\|D_t U\|_{H^{s-\frac{1}{2}}} + \|(V, B)\|_{H^s} \\
& \quad + \|\nabla_{x,z} \tilde{v}\|_{L_z^2(I; H^{s-\frac{1}{2}}}) + \|\eta\|_{H^{s+\frac{1}{2}}}).
\end{aligned}$$

Summing up the above estimates, we conclude the lemma. \square

9.5. Energy functional. We introduce an energy functional $\mathcal{E}_s(t)$ defined by

$$\begin{aligned}
\mathcal{E}_s(t) & \stackrel{\text{def}}{=} \|v(t)\|_{H^1(\Omega(t))} + \|\eta(t)\|_{H^s} + \|(V, B)(t)\|_{H^{s-\frac{1}{2}}} + \|V_b(t)\|_{H^s} \\
& \quad + \|\tilde{\omega}(t)\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})} + \|T_{\sqrt{a\lambda}} U(t)\|_{H^{s-\frac{1}{2}}} + \|D_t U(t)\|_{H^{s-\frac{1}{2}}}.
\end{aligned} \tag{9.10}$$

Proposition 9.15. *It holds that*

$$\|U\|_{H^s} + \|(V, B)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} + \|\tilde{v}\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})} \leq K_\eta A(t)^3 \mathcal{E}_s(t).$$

Proof. It follows from Lemma 8.1 that

$$\|\tilde{v}\|_{H^{s+\frac{1}{2}}(\bar{\mathcal{S}})} \leq K_\eta A(t) (\|(V, B, V_b)\|_{H^s} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}(\bar{\mathcal{S}})} + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

This together with Lemma 9.16 and Lemma 9.17 yields the result. \square

Lemma 9.16. *It holds that*

$$\|\eta\|_{H^{s+\frac{1}{2}}} \leq K_\eta A(t)^2 \mathcal{E}_s(t).$$

Proof. Recall that $D_t U = -T_a \zeta + h_1 + [D_t, T_\zeta]B$. Due to $a \geq c_0$, we have

$$\begin{aligned} \zeta &= T_{a-1} T_a \zeta + (T_{a-1} T_a - 1) \zeta \\ &= T_{a-1} (-D_t U + h_1 + [D_t, T_\zeta]B) + (T_{a-1} T_a - 1) \zeta \end{aligned}$$

which along with Proposition 2.6, Lemma 9.7, Proposition 2.21 and Lemma 7.8 yields

$$\|\zeta\|_{H^{s-\frac{1}{2}}} \leq K_\eta \|D_t U\|_{H^{s-\frac{1}{2}}} + K_\eta A(t)^2 (\|\eta\|_{H^s} + \|(V, B)\|_{H^{s-\frac{1}{2}}}).$$

This gives the lemma by recalling $\zeta = \nabla \eta$. \square

Lemma 9.17. *It holds that*

$$\|U\|_{H^s} + \|(V, B)\|_{H^s} \leq K_\eta A(t)^3 \mathcal{E}_s(t).$$

Proof. Recall that $U = V + T_\zeta B$. Hence,

$$\|V\|_{H^s} \leq \|U\|_{H^s} + \|T_\zeta B\|_{H^s} \leq K_\eta (\|U\|_{H^s} + \|B\|_{H^s}).$$

So, it is sufficient to consider B . We have

$$\operatorname{div} U = \operatorname{div} V + T_\zeta \cdot \nabla B + T_{\operatorname{div} \zeta} B.$$

On the other hand, we have

$$\begin{aligned} \operatorname{div} V &= \sum_{i=1}^d \partial_i v^i + \partial_i \eta \partial_y v^i \big|_{y=\eta} \\ &= -\partial_y v^{d+1} + \nabla \eta \cdot \nabla v^{d+1} \big|_{y=\eta} + \partial_i \eta \omega_{d+1,i} \big|_{y=\eta} \\ &= -G(\eta)B - \partial_y v_\omega^{d+1} + \nabla \eta \cdot \nabla v_\omega^{d+1} \big|_{y=\eta} + \partial_i \eta \omega_{d+1,i} \big|_{y=\eta} \\ &\triangleq -G(\eta)B + V_\omega. \end{aligned}$$

Then we deduce that

$$\begin{aligned} \operatorname{div} U &= \operatorname{div} V + T_\zeta \cdot \nabla B + T_{\operatorname{div} \zeta} B \\ &= -G(\eta)B + V_\omega + T_\zeta \cdot \nabla B + T_{\operatorname{div} \zeta} B \\ &= -T_\lambda B - R(\eta)B + V_\omega + T_\zeta \cdot \nabla B + T_{\operatorname{div} \zeta} B \\ &= -T_q B - R(\eta)B + V_\omega + T_{\operatorname{div} \zeta} B, \end{aligned}$$

where the symbol $q = \lambda - i\zeta \cdot \xi$. Thus, it follows from Proposition 8.3, Lemma 8.4 and Lemma 9.16 that

$$\begin{aligned} \|T_q B\|_{H^{s-1}} &\leq \|U\|_{H^s} + \|R(\eta)B\|_{H^{s-1}} + \|V_\omega\|_{H^{s-1}} + K_\eta \|B\|_{H^{s-1}} \\ &\leq \|U\|_{H^s} + K_\eta A(t) (\|(B, V_b)\|_{H^{s-1}} + \|\omega\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} + \|\eta\|_{H^{s+\frac{1}{2}}}) \\ &\leq \|U\|_{H^s} + K_\eta A(t)^3 \mathcal{E}(t). \end{aligned}$$

On the other hand, we get by Proposition 2.6 and Lemma 7.8 that

$$\begin{aligned} \|U\|_{H^s} &\leq \|T_{(\sqrt{a\lambda})^{-1}} T_{\sqrt{a\lambda}} U\|_{H^s} + \|(T_{(\sqrt{a\lambda})^{-1}} T_{\sqrt{a\lambda}} - 1)U\|_{H^s} \\ &\leq K_\eta A(t)^2 \|T_{\sqrt{a\lambda}} U\|_{H^{s-\frac{1}{2}}} + K_\eta A(t)^2 \|U\|_{H^{s-\frac{1}{2}}} \\ &\leq K_\eta A(t)^2 \|T_{\sqrt{a\lambda}} U\|_{H^{s-\frac{1}{2}}} + K_\eta A(t)^2 \|(V, B)\|_{H^{s-\frac{1}{2}}}, \end{aligned}$$

from which and Proposition 2.6, we infer that

$$\begin{aligned} \|B\|_{H^s} &\leq K_\eta \|T_q B\|_{H^{s-1}} + \|(T_{q^{-1}} T_q - 1)B\|_{H^s} \\ &\leq K_\eta A(t)^3 \mathcal{E}_s(t) + K_\eta \|B\|_{H^{s-\frac{1}{2}}} \leq K_\eta A(t)^3 \mathcal{E}_s(t). \end{aligned}$$

The proof is finished. \square

9.6. Proof of Theorem 1.3. We first recover the regularity of the free surface from the mean curvature.

Lemma 9.18. *Assume that the mean curvature $\kappa \in L^2 \cap L^p(\mathbf{R}^d)$ for some $p > d$. Then we have*

$$\|\eta\|_{H^2} + \|\eta\|_{C^{2-\frac{d}{p}}} \leq C(\|\eta\|_{L^2}, \|\nabla \eta\|_{L^\infty}, \|H\|_{L^2 \cap L^p}).$$

Proof. The estimate of $\|\eta\|_{C^{2-\frac{d}{p}}}$ has been proved in [32]. Let $\eta_\ell = \partial_\ell \eta$ and $a_{ij} = (1 + |\nabla \eta|^2)^{-\frac{3}{2}}((1 + |\nabla \eta|^2)\delta_{ij} - \partial_i \eta \partial_j \eta)$. A direct calculation gives

$$\partial_j(a_{ij} \partial_i \eta_\ell) = \partial_\ell H.$$

It is easy to verify that the matrix (a_{ij}) is uniformly elliptic with the elliptic constants depending on $\|\nabla \eta\|_{L^\infty}$, which implies that $\|\nabla \eta_\ell\|_{L^2} \leq C(\|\nabla \eta\|_{L^\infty})\|H\|_{L^2}$. \square

Lemma 9.19. *It holds that*

$$\sup_{t \in [0, T]} (A(t) + \|\eta(t)\|_{C^{\frac{3}{2}+\varepsilon}}) \leq C(T, M(T), \|v_0\|_{H^1(\Omega_0)}, \|\eta_0\|_{H^s}),$$

where C is an increasing function depending on h_0 .

Proof. Recall that $\zeta = \nabla \eta$ satisfies

$$\partial_t \zeta + V \cdot \nabla \zeta = \nabla B + \nabla V \cdot \zeta,$$

which implies

$$\sup_{t \in [0, T]} \|\nabla \eta(t)\|_{L^\infty} \leq C(T, M(T), \|\nabla \eta_0\|_{L^\infty}),$$

which along with Lemma 9.1 and Lemma 9.18 implies that

$$\sup_{t \in [0, T]} (\|\eta(t)\|_{H^2} + \|\eta(t)\|_{C^{\frac{3}{2}+\varepsilon}}) \leq C(T, M(T), \|v_0\|_{L^2(\Omega_0)}, \|\eta_0\|_{H^s}). \quad (9.11)$$

By Lemma 9.1 and Lemma 9.2, we have

$$\frac{d}{dt} (\|v\|_{H^1(\Omega_t)}^2 + \|\eta\|_{L^2}^2) \leq K_\eta (\|v(t)\|_{W^{1,\infty}(\Omega_t)} \|v(t)\|_{H^1(\Omega_t)} + \|\eta(t)\|_{H^{\frac{3}{2}}}),$$

from which and (9.11), we deduce the lemma. \square

Lemma 9.20. *Let $s > \frac{d}{2} + 1$ and $f \in H^s(\mathbf{R}^d)$. Then there holds*

$$\|f\|_{B_{\infty,1}^1} \leq C(1 + \|f\|_{W^{1,\infty}}) \ln(e + \|f\|_{H^s}).$$

Proof. Given an integer N , we get by Lemma 2.2 that

$$\begin{aligned} \|f\|_{B_{\infty,1}^1} &= \sum_{j \geq -1}^N 2^j \|\Delta_j f\|_{L^\infty} + \sum_{j > N} 2^j \|\Delta_j f\|_{L^\infty} \\ &\leq C(N+1) \|f\|_{W^{1,\infty}} + C \sum_{j > N} 2^{(1+\frac{d}{2})j} \|\Delta_j f\|_{L^2} \\ &\leq C(N+1) \|f\|_{W^{1,\infty}} + C 2^{-N(s-1-\frac{d}{2})} \|f\|_{H^s}. \end{aligned}$$

Take N so that $2^{-N(s-1-\frac{d}{2})} \|f\|_{H^s} \sim 1$ (i.e., $N \sim \ln(e + \|f\|_{H^s})$). Then the lemma follows easily. \square

Now we are in position to prove Theorem 1.3. We denote

$$\mathcal{P}_T \triangleq \mathcal{P}(T, M(T), \|\nabla \eta_0\|_{L^\infty})$$

for some increasing function \mathcal{P} depending on c_0, h_0 , which may change from line to line. By Lemma 9.19, K_η and $A(t)$ is bounded by \mathcal{P}_T . By Proposition 9.15, we have

$$\|U\|_{H^s} + \|(V, B)\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} + \|\tilde{v}\|_{H^{s+\frac{1}{2}}(\overline{\mathcal{S}})} \leq P_T \mathcal{E}_s(t).$$

We first deduce from Proposition 9.4 that

$$\frac{d}{dt} (\|D_t U\|_{H^{s-\frac{1}{2}}}^2 + \|T_{\sqrt{a\lambda}} U\|_{H^{s-\frac{1}{2}}}^2) \leq \mathcal{P}_T \mathcal{B}(t) \mathcal{E}_s(t)^2 + 2 \langle (f + f_\omega)_{s-1/2}, (D_t U)_{s-1/2} \rangle,$$

which along with Lemma 9.10, Lemma 9.13 and Lemma 9.14 gives

$$\begin{aligned} \frac{d}{dt} \left(\|D_t U\|_{H^{s-\frac{1}{2}}}^2 + \|T_{\sqrt{a\lambda}} U\|_{H^{s-\frac{1}{2}}}^2 + \langle (f_\omega)_{s-1}, U_s \rangle \right. \\ \left. + \langle g_{s-1/2}, h_{s-1/2} \rangle + \frac{1}{2} \langle g_{s-1/2}, g_{s-1/2} \rangle \right) \leq \mathcal{P}_T \mathcal{B}(t) \mathcal{E}_s(t)^2, \end{aligned}$$

where $g = [D_t, T_\zeta]B$ and $h = D_t U - [D_t, T_\zeta]B$. By the proof of Lemma 9.14 and Lemma 9.11, we know

$$\begin{aligned} \langle (f_\omega)_{s-1}, U_s \rangle &\leq K_\eta A(t)^3 (\|V_b\|_{H^{\frac{1}{2}}} + \|\tilde{\omega}\|_{H^{s-\frac{1}{2}}} + \|\eta\|_{H^s}) \|U\|_{H^s}, \\ \langle g_{s-1/2}, h_{s-1/2} \rangle &\leq K_\eta A(t) \|B\|_{H^{s-\frac{1}{2}}} \|D_t U\|_{H^{s-\frac{1}{2}}} + K_\eta A(t)^3 \|B\|_{H^{s-\frac{1}{2}}}^2, \\ \langle g_{s-1/2}, g_{s-1/2} \rangle &\leq K_\eta A(t)^2 \|B\|_{H^{s-\frac{1}{2}}}^2. \end{aligned}$$

This shows that

$$\begin{aligned} \|D_t U(t)\|_{H^{s-\frac{1}{2}}} + \|T_{\sqrt{a\lambda}} U(t)\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{P}_T \mathcal{E}_s(0) + \mathcal{P}_0 \int_0^t \mathcal{B}(t') \mathcal{E}_s(t') dt' \\ &\quad + \mathcal{P}_T (\|(V, B)(t)\|_{H^{s-\frac{1}{2}}} + \|V_b(t)\|_{H^s} + \|\tilde{\omega}(t)\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} + \|\eta(t)\|_{H^s}), \end{aligned}$$

which together with Proposition 9.5, Proposition 9.3 and Lemma 9.20 gives

$$\begin{aligned} \mathcal{E}_s(t) &\leq \mathcal{P}_0 \mathcal{E}_s(0) + \mathcal{P}_T \int_0^t \mathcal{B}(t') \mathcal{E}_s(t') dt' \\ &\leq \mathcal{P}_T \mathcal{E}_s(0) + \mathcal{P}_T \int_0^t \ln(e + \mathcal{E}_s(t')) \mathcal{E}_s(t') dt'. \end{aligned}$$

Then Gronwall's inequality ensures that

$$\mathcal{E}_s(t) \leq C(E_s(0), \mathcal{P}_T).$$

This completes the proof of Theorem 1.3. \square

10. ITERATION SCHEME AND SYMMETRIZATION

We begin with the proof of local well-posedness from this section. We first establish the local well-posedness result for sufficiently smooth data. In the last section, we extend it to the low regularity data. Although there has been a lot of work [26, 31, 39] devoted to the local well-posedness of the Euler equations with free surface for smooth data, they do not work in the Eulerian coordinates and do not consider the case of finite depth to our knowledge.

Theorem 10.1. *Let $d \geq 1$ and $s > \frac{d}{2} + 10$ be an integer. Assume that the initial data (η_0, v_0) satisfies*

$$\eta_0 \in H^{s+1/2}(\mathbf{R}^d), \quad v_0 \in H^s(\Omega_0).$$

Furthermore, we assume that there exist two positive constants $c_0 > 0$ and $h_0 > 0$ such that

$$\begin{aligned} -(\partial_y P)(0, x, \eta_0(x)) &\geq c_0 \quad \text{for } x \in \mathbf{R}^d, \\ 1 + \eta_0(x) &\geq h_0 \quad \text{for } x \in \mathbf{R}^d. \end{aligned}$$

Then there exists $T > 0$ such that the system (1.1)–(1.5) with the initial data (η_0, v_0) has a unique solution (η, v) satisfying

$$\eta \in C([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d)), \quad v \in C([0, T]; H^s(\Omega_t)).$$

The proof of Theorem 10.1 is conducted in the following four section. In this section, we construct a sequence of approximate solutions by an iteration. The next section is devoted to the uniform estimates of the approximate solutions. The last two sections are devoted to show that the approximate sequence is a Cauchy sequence and converges to the solution of the Euler system (1.1)–(1.5).

10.1. Iteration scheme. We construct the approximate solution by an iteration. Assume that the initial data (η_0, v_0) is smooth. Assume that we construct a smooth solution $(V^n, B^n, V_b^n, \eta^n, P^n, \omega^n)$ and $(V_1^n, B_1^n, V_{b,1}^n, \eta_1^n)$ in n -th iteration. Here ω^n is a function defined in $\Omega_t^n = \{(x, y) : -1 < y < \eta_1^n(t, x)\}$ and P^n is a function defined in $\{(x, y) : -1 < y < \eta^n(t, x)\}$. We will construct the solution $(V^{n+1}, B^{n+1}, V_b^{n+1}, \eta^{n+1}, P^{n+1}, \omega^{n+1})$ and $(V_1^{n+1}, B_1^{n+1}, V_{b,1}^{n+1}, \eta_1^{n+1})$ in the $(n+1)$ -th iteration by the following scheme.

We still denote by $D_t \triangleq \partial_t + T_{V^n} \cdot \nabla$ for the simplicity of notation. Let

$$a^n = -\partial_y P^n|_{y=\eta^n}, \quad \lambda^n = \lambda(\eta_1^n), \quad \zeta^n = \nabla \eta^n, \quad \zeta_1^n = \nabla \eta_1^n.$$

We first introduce the evolutionary system on the trace of velocity and the free surface in the $(n+1)$ -th iteration.

$$\begin{cases} D_t V^{n+1} = -T_{\zeta^n} a^n - T_{a^n} \zeta^{n+1} - R(\zeta^n, a^n) + (T_{V_1^n} - V_1^n) \cdot \nabla V^n, \\ D_t B^{n+1} = a^n - 1 + (T_{V_1^n} - V_1^n) \cdot \nabla B^n, \\ (\partial_t + V_b^n \cdot \nabla) V_b^{n+1} = -\nabla P^n|_{y=-1}, \\ D_t \zeta^{n+1} = T_{\lambda^n} (V^{n+1} + T_{\zeta^n} B^{n+1}) + (T_{V_1^n} - V_1^n) \cdot \zeta^n + [T_{\zeta^n}, T_{\lambda^n}] B_1^n \\ \quad + (\zeta^n - T_{\zeta^n}) T_{\lambda^n} B^n + R(\eta_1^n) V_1^n + \zeta^n R(\eta_1^n) B_1^n + R_\omega^n, \\ (V^{n+1}, B^{n+1}, \zeta^{n+1}, V_b^{n+1})|_{t=0} = (V_0, B_0, \nabla \eta_0, V_{b,0}), \end{cases} \quad (10.1)$$

where R_ω^n is given by

$$\begin{aligned} (R_\omega^n)^i &= (\partial_y (v_\omega^n)^i - \partial_{x_j} (v_\omega^n)^i \cdot \partial_{x_j} \eta_1^n) + \partial_{x_i} \eta_1^n (\partial_y (v_\omega^n)^{d+1} - \partial_{x_j} \eta_1^n \partial_{x_j} (v_\omega^n)^{d+1}) \\ &\quad + (\omega_{i,d+1}^n - \partial_{x_j} \eta_1^n \omega_{ij}^n + \partial_{x_i} \eta_1^n \partial_{x_j} \eta_1^n \omega_{j,d+1}^n)|_{y=\eta_1^n}. \end{aligned}$$

Here v_ω^n solves the following elliptic equation in Ω_t^n :

$$\begin{cases} -\Delta_{x,y} v_\omega^n = \nabla_{x,y} \times \omega^n & \text{in } \Omega_t^n, \\ v_\omega^n|_{y=\eta_1^n} = 0, \quad v_\omega^n|_{y=-1} = (V_{b,1}^n, 0). \end{cases}$$

Given $(V^{n+1}, B^{n+1}, V_b^{n+1})$, we introduce a new boundary velocity $(V_1^{n+1}, B_1^{n+1}, V_{b,1}^{n+1})$ defined by

$$\begin{cases} (\partial_t + T_{V^{n+1}} \cdot \nabla) (V_1^{n+1}, B_1^{n+1}) = D_t (V^{n+1}, B^{n+1}), \\ (\partial_t + V_b^{n+1} \cdot \nabla) V_{b,1}^{n+1} = -\nabla P^n|_{y=-1}, \\ (V_1^{n+1}, B_1^{n+1}, V_{b,1}^{n+1})|_{t=0} = (V_0, B_0, V_{b,0}). \end{cases} \quad (10.2)$$

A key property is that (V_1^{n+1}, B_1^{n+1}) has the same regularity as $(\partial_t + T_{V^{n+1}} \cdot \nabla) (V_1^{n+1}, B_1^{n+1})$. While, $(\partial_t + T_{V^{n+1}} \cdot \nabla) (V^{n+1}, B^{n+1})$ will lose one derivative.

Given $(V^{n+1}, B^{n+1}, \zeta^{n+1})$, we define η^{n+1} by

$$-\Delta \eta^{n+1} + \eta^{n+1} = -\operatorname{div} \zeta^{n+1} + \eta_1^{n+1}, \quad (10.3)$$

where η_1^{n+1} is determined by

$$\begin{cases} (\partial_t + V^{n+1} \cdot \nabla) \eta_1^{n+1} = B_1^{n+1} \\ \eta_1^{n+1}|_{t=0} = \eta_0. \end{cases} \quad (10.4)$$

Given η_1^{n+1} , let $\Omega_t^{n+1} = \{(x, y) : -1 \leq y \leq \eta_1^{n+1}(t, x)\}$. The vorticity ω^{n+1} in $(n+1)$ -th iterative is given by solving the nonlinear vorticity equation in the known domain Ω_t^{n+1} :

$$\begin{cases} \partial_t \omega^{n+1} + ((v^{n+1})^h \cdot \nabla + (v_1^{n+1})^{d+1} \partial_y) \omega^{n+1} = \omega^{n+1} \cdot \nabla_{x,y} v_1^{n+1}, \\ \omega^{n+1}|_{t=0} = \omega_0, \end{cases} \quad (10.5)$$

where the velocity v^{n+1} is given by

$$\begin{cases} -\Delta v^{n+1} = \nabla_{x,y} \times \omega^{n+1} & \text{in } \Omega_t^{n+1}, \\ v^{n+1}|_{y=\eta_1^{n+1}} = (V^{n+1}, B^{n+1}), \\ v^{n+1}|_{y=-1} = (V_b^{n+1}, 0), \end{cases} \quad (10.6)$$

and the velocity v_1^{n+1} is given by

$$\begin{cases} -\Delta v_1^{n+1} = \nabla_{x,y} \cdot \omega^{n+1} & \text{in } \Omega_t^{n+1}, \\ v_1^{n+1}|_{y=\eta_1^{n+1}} = (V_1^{n+1}, B_1^{n+1}), \\ v_1^{n+1}|_{\Gamma_b} = (V_{b,1}^{n+1}, 0). \end{cases} \quad (10.7)$$

Finally, we need to construct the pressure P^{n+1} in a smoother domain $\tilde{\Omega}_t^{n+1} = \{(x, y) : -1 < y < \eta^{n+1}(t, x)\}$:

$$\begin{cases} -\Delta P^{n+1} = (\partial_i (v_1^j)^{n+1} \partial_j (v_1^i)^{n+1}) \circ \Phi_1^{n+1} \circ (\Phi^{n+1})^{-1}, \\ P^{n+1}|_{y=\eta^{n+1}} = 0, \quad (\partial_y P^{n+1})|_{y=-1} = -1, \end{cases} \quad (10.8)$$

where the map Φ^{n+1} and Φ_1^{n+1} are given by

$$\begin{aligned} \Phi^{n+1} : (x, z) \in \mathcal{S} &\longmapsto (x, \rho_\delta^{n+1}(x, z)) \in \tilde{\Omega}_t^{n+1}, \\ \Phi_1^{n+1} : (x, z) \in \mathcal{S} &\longmapsto (x, \rho_{\delta,1}^{n+1}(x, z)) \in \Omega_t^{n+1}. \end{aligned}$$

Here $\rho_\delta^{n+1} = \rho_{\delta, \eta^{n+1}}$ and $\rho_{\delta,1}^{n+1} = \rho_{\delta, \eta_1^{n+1}}$ with $\rho_{\delta, \eta}(x, z) = z + (1+z)e^{\delta z|D|\eta}$.

10.2. Symmetrization. We introduce a good unknown $U^{n+1} = V^{n+1} + T_{\zeta^n} B^{n+1}$. It follows from (10.1) that

$$\begin{cases} D_t U^{n+1} + T_{a^n} \zeta^{n+1} = h_1^n + [D_t, T_{\zeta^n}] B^{n+1}, \\ D_t \zeta^{n+1} = T_{\lambda^n} U^{n+1} + h_2^n + R_\omega^n, \end{cases} \quad (10.9)$$

where (h_1^n, h_2^n) is given by

$$\begin{aligned} h_1^n &= (T_{V_1^n} - V_1^n) \cdot \nabla V^n - R(a^n, \zeta^n) + T_{\zeta^n} (T_{V_1^n} - V_1^n) \cdot \nabla B^n, \\ h_2^n &= (T_{V_1^n} - V_1^n) \cdot \nabla \zeta^n + [T_{\zeta^n}, T_{\lambda^n}] B_1^n + (\zeta^n - T_{\zeta^n}) T_{\lambda^n} B^n \\ &\quad + R(\eta_1^n) V_1^n + \zeta^n R(\eta_1^n) B_1^n. \end{aligned}$$

A direct calculation gives

$$D_t^2 \zeta^{n+1} + T_{a^n \lambda^n} \zeta^{n+1} = f_1^n + f_2^n + D_t f_3^n + D_t f_4^n + D_t f_\omega^n, \quad (10.10)$$

where $(f_1^n, f_2^n, f_3^n, f_4^n, f_\omega^n)$ are given by

$$\begin{aligned} f_1^n &= D_t ((T_{V_1^n} - V_1^n) \cdot \nabla \zeta^n + [T_{\zeta^n}, T_{\lambda^n}] B_1^n) + (T_{a^n \lambda^n} - T_{\lambda^n} T_{a^n}) \zeta^{n+1}, \\ f_2^n &= [D_t, T_{\lambda^n}] U^{n+1} + T_{\lambda^n} (h_1^n + [D_t, T_{\zeta^n}] B^{n+1}), \\ f_3^n &= (\zeta^n - T_{\zeta^n}) T_{\lambda^n} B^n + (\zeta^n - T_{\zeta^n}) R(\eta_1^n) B_1^n, \\ f_4^n &= R(\eta_1^n) V_1^n + T_{\zeta^n} R(\eta_1^n) B_1^n, \\ f_\omega^n &= R_\omega^n. \end{aligned}$$

The local existence of smooth solution for the approximate system (10.1)–(10.8) can be proved by using the theory of symmetric hyperbolic system and elliptic equations. Here we ignore the proof.

11. UNIFORM ENERGY ESTIMATES

11.1. **Set-up.** Throughout this section, we denote

$$D_t \triangleq \partial_t + T_{V^n} \cdot \nabla, \quad D_t^u \triangleq \partial_t + V^n \cdot \nabla, \quad D_t^b \triangleq \partial_t + V_b^n \cdot \nabla, \quad \mathcal{D}_t \triangleq \partial_t + v_2^n \cdot \nabla_{x,y},$$

where $v_2^n = ((v^n)^h, (v_1^n)^{d+1})$. We denote

$$\tilde{D}_t \triangleq \partial_t + T_{V^{n+1}} \cdot \nabla, \quad \tilde{D}_t^u \triangleq \partial_t + V^{n+1} \cdot \nabla, \quad \tilde{D}_t^b \triangleq \partial_t + V_b^{n+1} \cdot \nabla, \quad \tilde{\mathcal{D}}_t \triangleq \partial_t + v_2^{n+1} \cdot \nabla_{x,y},$$

where $v_2^{n+1} = ((v^{n+1})^h, (v_1^{n+1})^{d+1})$. For a function $f(x, y)$ defined on $\{(x, y) : -1 < y < \eta(x)\}$, we denote $\tilde{f}(x, z) \triangleq f(x, \rho_\delta(x, z))$, where $\rho_\delta(x, z) = z + (1 + z)e^{\delta z|D|}\eta$.

Let $E_i (i = 1, \dots, 6)$ be some constants determined later. Assume that there exists T independent of n determined later such that the solution in the n -th iteration satisfies

H1. For any $t \in [0, T]$, there holds

$$\|(V^n, B^n, V_b^n, V_1^n, B_1^n, V_{b,1}^n)(t)\|_{H^{s-\frac{1}{2}}} + \|(\eta^n, \eta_1^n)(t)\|_{H^{s-\frac{1}{2}}} + \|\tilde{\omega}^n(t)\|_{H^{s-1}(\overline{\mathcal{S}})} \leq E_1;$$

H2. For any $t \in [0, T]$, there holds

$$\|\nabla_{x,z} \tilde{v}_1^n(t)\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\tilde{v}_\omega^n(t)\|_{H^s(\overline{\mathcal{S}})} + \|a^n(t)\|_{H^{s-\frac{3}{2}}} + \|\nabla P^n(t, \cdot, -1)\|_{H^{s-\frac{3}{2}}} \leq E_2;$$

H3. For any $t \in [0, T]$, there holds

$$\|a^n(t)\|_{H^{s-\frac{1}{2}}} + \|\nabla P^n(t, \cdot, -1)\|_{H^{s-\frac{1}{2}}} \leq E_3, \quad \|\eta^n(t)\|_{H^{s+\frac{1}{2}}} \leq E_3;$$

H4. For any $t \in [0, T]$, there holds

$$\begin{aligned} & \|D_t(V_1^n, B_1^n)(t)\|_{H^{s-\frac{3}{2}}} + \|\partial_t \eta^n(t)\|_{H^{s-\frac{3}{2}}} + \|D_t^u \eta_1^n(t)\|_{H^{s-\frac{1}{2}}} \\ & + \|\widetilde{\mathcal{D}_t \omega^n}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\widetilde{\mathcal{D}_t v_\omega^n}(t)\|_{X^{s-\frac{1}{2}}([-\frac{1}{2}, 0])} \leq E_4; \end{aligned}$$

H5. For any $t \in [0, T]$, there holds

$$\begin{aligned} & \|(D_t(V_1^n, B_1^n), D_t^b V_{b,1}^n)(t)\|_{H^{s-\frac{1}{2}}} + \|\partial_t^2 \eta^n(t)\|_{H^{s-\frac{5}{2}}} + \|\partial_t a^n(t)\|_{H^{s-\frac{3}{2}}} \\ & + \|(D_t^u)^2 \eta_1^n(t)\|_{H^{s-\frac{1}{2}}} + \|\widetilde{\mathcal{D}_t^2 \omega^n}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\nabla_{x,z} \widetilde{\mathcal{D}_t v_1^n}\|_{H^{s-1}(\overline{\mathcal{S}})} \\ & + \|\nabla \partial_t P^n(t, \cdot, -1)\|_{H^{s-\frac{3}{2}}} \leq E_5; \end{aligned}$$

H6. For any $t \in [0, T]$, there holds

$$\|\partial_t (D_t(V_1^n, B_1^n), D_t^b V_{b,1}^n)(t)\|_{H^{s-\frac{3}{2}}} + \|\widetilde{\mathcal{D}_t^2 v_\omega^n}(t)\|_{X^{s-\frac{3}{2}}([-\frac{1}{2}, 0])} \leq E_6;$$

H7. For any $(t, x) \in [0, T] \times \mathbf{R}^d$, there holds

$$a^n(t, x) = -\partial_y P^n|_{y=\eta^n(t,x)} \geq \frac{c_0}{2};$$

H8. For any $(t, x) \in [0, T] \times \mathbf{R}^d$, there holds

$$\eta^n(t, x) + 1 \geq \frac{h_0}{2}, \quad \eta_1^n(t, x) + 1 \geq \frac{h_0}{2}.$$

The purpose of this section is to show that (H1)–(H8) also hold for the solution in the $(n+1)$ -th iteration.

In the sequel, we denote $\mathcal{A}_k = \mathcal{A}_k(E_1, \dots, E_k)$ for $k = 1, \dots, 6$, and $P_{s, \eta_1, \dots, \eta_k} = P_{s, \eta_1, \dots, \eta_k}(\|\eta_1\|_{H^s}, \dots, \|\eta_k\|_{H^s})$, where \mathcal{A}_k and $P_{s, \eta_1, \dots, \eta_k}$ are some nondecreasing functions depending on c_0, h_0 and may change from line to line. We also denote by

$\mathcal{P}(\cdot, \dots, \cdot)$ some increasing function depending on c_0, h_0 , which may be different from line to line.

11.2. Energy functional. We introduce the energy functional $\mathcal{E}^{n+1}(t)$ defined by

$$\mathcal{E}^{n+1}(t) \stackrel{\text{def}}{=} \mathcal{E}_1^{n+1}(t) + \mathcal{E}_2^{n+1}(t),$$

where $\mathcal{E}_1^{n+1}(t)$ and $\mathcal{E}_2^{n+1}(t)$ are given by

$$\begin{aligned} \mathcal{E}_1^{n+1}(t) &= \|D_t \zeta^{n+1}\|_{H^{s-1}} + \|T_{\sqrt{a^n \lambda^n}} \zeta^{n+1}\|_{H^{s-1}}, \\ \mathcal{E}_2^{n+1}(t) &= \|\tilde{\omega}^{n+1}\|_{H^{s-1}(\overline{S})} + \|\zeta^{n+1}\|_{H^{s-\frac{3}{2}}} + \|(V^{n+1}, B^{n+1}, V_b^{n+1})\|_{H^{s-\frac{1}{2}}} \\ &\quad + \|(V_1^{n+1}, B_1^{n+1}, V_{b,1}^{n+1}, \eta_1^{n+1})\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

Using the equations, we can establish the following regularity information in terms of $\mathcal{E}^{n+1}(t)$ for the solution in the $(n+1)$ -th iteration.

Lemma 11.1. *It holds that*

$$\|\eta^{n+1}\|_{H^{s+\frac{1}{2}}} \leq \mathcal{A}_2 \mathcal{E}^{n+1}(t), \quad \|U^{n+1}\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_1 \mathcal{E}_2^{n+1}(t).$$

Proof. We write

$$\zeta^{n+1} = T_{(\sqrt{a^n \lambda^n})^{-1}} T_{\sqrt{a^n \lambda^n}} \zeta^{n+1} + (T_{(\sqrt{a^n \lambda^n})^{-1}} T_{\sqrt{a^n \lambda^n}} - 1) \zeta^{n+1},$$

from which and Proposition 2.6, we infer that

$$\begin{aligned} \|\zeta^{n+1}\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{A}_2 \|\zeta^{n+1}\|_{H^{s-\frac{3}{2}}} + \mathcal{A}_2 \|T_{\sqrt{a^n \lambda^n}} \zeta^{n+1}\|_{H^{s-1}} \\ &\leq \mathcal{A}_2 \mathcal{E}^{n+1}(t). \end{aligned}$$

Recall that η^{n+1} satisfies

$$-\Delta \eta^{n+1} + \eta^{n+1} = -\operatorname{div} \zeta^{n+1} + \eta_1^{n+1},$$

which implies that

$$\|\eta^{n+1}\|_{H^{s+\frac{1}{2}}} \leq \|\zeta^{n+1}\|_{H^{s-\frac{1}{2}}} + \|\eta_1^{n+1}\|_{H^{s-\frac{3}{2}}} \leq \mathcal{A}_2 \mathcal{E}^{n+1}(t).$$

The second inequality follows from $U^{n+1} = V^{n+1} + T_{\zeta^n} B^{n+1}$. \square

Lemma 11.2. *It holds that*

$$\|D_t U^{n+1}\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_1 + \mathcal{A}_4 \mathcal{E}^{n+1}(t).$$

Proof. Recall that

$$D_t U^{n+1} + T_{a^n} \zeta^{n+1} = h_1^n + [D_t, T_{\zeta^n}] B^{n+1}. \quad (11.1)$$

By Lemma 2.10, we have

$$\|h_1^n\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_1,$$

which along with Proposition 2.21 and Lemma 11.1 gives

$$\begin{aligned} \|D_t U^{n+1}\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{A}_1 + \mathcal{A}_2 \|\zeta^{n+1}\|_{H^{s-\frac{1}{2}}} + \mathcal{A}_4 \|B^{n+1}\|_{H^{s-\frac{1}{2}}} \\ &\leq \mathcal{A}_1 + \mathcal{A}_4 \mathcal{E}^{n+1}(t). \end{aligned}$$

The proof is finished. \square

Lemma 11.3. *It holds that*

$$\begin{aligned}\|\tilde{D}_t^u \eta_1^{n+1}\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{E}_2^{n+1}(t), \\ \|(\tilde{D}_t^u)^2 \eta_1^{n+1}\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{A}_3(1 + \mathcal{E}_2^{n+1}(t))^2, \\ \|D_t \zeta^{n+1}\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_2(1 + \mathcal{E}_2^{n+1}(t)), \\ \|\partial_t \eta^{n+1}\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_2(1 + \mathcal{E}_2^{n+1}(t))^2, \\ \|\partial_t^2 \eta^{n+1}\|_{H^{s-\frac{5}{2}}} &\leq \mathcal{A}_4(1 + \mathcal{E}_2^{n+1}(t))^3.\end{aligned}$$

Proof. The first two inequalities are obvious. Recall that

$$\begin{aligned}D_t \zeta^{n+1} &= T_{\lambda^n}(V^{n+1} + T_{\zeta^n} B^{n+1}) + (T_{V_1^n} - V_1^n) \cdot \zeta^n + [T_{\zeta^n}, T_{\lambda^n}] B_1^n \\ &\quad + (\zeta^n - T_{\zeta^n}) T_{\lambda^n} B^n + R(\eta_1^n) V_1^n + \zeta^n R(\eta_1^n) B_1^n + R_\omega^n.\end{aligned}$$

Then it follows from Proposition 2.6 and Proposition 5.2 that

$$\|D_t \zeta^{n+1}\|_{H^{s-\frac{3}{2}}} \leq \mathcal{A}_1 \mathcal{E}_2^{n+1}(t) + \mathcal{A}_2.$$

For $\partial_t \eta^{n+1}$, we get by the elliptic estimate that

$$\begin{aligned}\|\partial_t \eta^{n+1}\|_{H^{s-\frac{3}{2}}} &\leq C(\|\partial_t \zeta^{n+1}\|_{H^{s-\frac{5}{2}}} + \|\partial_t \eta_1^{n+1}\|_{H^{s-\frac{7}{2}}}) \\ &\leq \mathcal{A}_2 + \mathcal{A}_1 \mathcal{E}_2^{n+1}(t) + C \mathcal{E}_2^{n+1}(t)^2.\end{aligned}$$

Using Proposition 2.6 and Proposition 11.7, we can also deduce that

$$\|\partial_t D_t \zeta^{n+1}\|_{H^{s-\frac{5}{2}}} \leq \mathcal{A}_4 \mathcal{E}_2^{n+1}(t) + \mathcal{A}_4,$$

which implies the estimate for $\partial_t^2 \eta^{n+1}$. \square

Lemma 11.4. *It holds that*

$$\begin{aligned}\|(D_t V^{n+1}, \tilde{D}_t V_1^{n+1})\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{A}_3(1 + \mathcal{E}^{n+1}(t)), \\ \|(D_t B^{n+1}, \tilde{D}_t B_1^{n+1}, \tilde{D}_t^b V_{b,1}^{n+1})\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{A}_3, \\ \|(D_t^b V_b^{n+1}, \tilde{D}_t^b V_{b,1}^{n+1})\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_2, \\ \|(\tilde{D}_t V_1^{n+1}, \tilde{D}_t B_1^{n+1})\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_2(1 + \mathcal{E}_2^{n+1}(t)), \\ \|(\partial_t \tilde{D}_t(V_1^{n+1}, B_1^{n+1}), \partial_t \tilde{D}_t^b V_{b,1}^{n+1})\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_5(1 + \mathcal{E}_2^{n+1}(t)).\end{aligned}$$

Proof. Recall that $D_t V^{n+1} = \tilde{D}_t V_1^{n+1}$ and

$$D_t V^{n+1} = -T_{\zeta^n} a^n - T_{a^n} \zeta^{n+1} - R(\zeta^n, a^n) + (T_{V_1^n} - V_1^n) \cdot \nabla V^n.$$

It follows from Lemma 2.10 and Lemma 11.1 that

$$\|(D_t V^{n+1}, \tilde{D}_t V_1^{n+1})\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_3 + \mathcal{A}_2 \|\zeta^{n+1}\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_3 + \mathcal{A}_2 \mathcal{E}^{n+1}(t).$$

The proof of the other inequalities is similar. \square

11.3. Estimate of the velocity. In the sequel, we assume that (H8) holds for $(\eta^{n+1}, \eta_1^{n+1})$, i.e.,

$$\eta^{n+1}(t, x) + 1 \geq \frac{h_0}{2}, \quad \eta_1^{n+1}(t, x) + 1 \geq \frac{h_0}{2}.$$

Recall that v^{n+1} satisfies

$$\begin{cases} -\Delta_{x,y} v^{n+1} = \nabla_{x,y} \times \omega^{n+1} & \text{in } \Omega_t^{n+1}, \\ v^{n+1}|_{y=\eta_1^{n+1}} = (V^{n+1}, B^{n+1}), \quad v^{n+1}|_{y=-1} = (V_b^{n+1}, 0), \end{cases}$$

and v_1^{n+1} satisfies

$$\begin{cases} -\Delta_{x,y} v_1^{n+1} = \nabla_{x,y} \times \omega^{n+1} & \text{in } \Omega_t^{n+1}, \\ v_1^{n+1}|_{y=\eta_1^{n+1}} = (V_1^{n+1}, B_1^{n+1}), \quad v_1^{n+1}|_{y=-1} = (V_{b,1}^{n+1}, 0). \end{cases}$$

First of all, we apply Proposition 4.10 to obtain

$$\begin{aligned} \|\nabla_{x,z} \widetilde{v^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})} &\leq P_{s-\frac{1}{2}, \eta_1^{n+1}}(\|(V^{n+1}, B^{n+1}, V_b^{n+1})\|_{H^{s-\frac{1}{2}}} + \|\widetilde{\omega^{n+1}}\|_{H^{s-1}}) \\ &\leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)). \end{aligned} \quad (11.2)$$

Similarly, we have

$$\|\nabla_{x,z} \widetilde{v_1^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})} \leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)). \quad (11.3)$$

A direct calculation gives

$$\begin{cases} \Delta_{x,y} \widetilde{\mathcal{D}_t v_1^{n+1}} = h_\omega^{n+1} & \text{in } \Omega_t^{n+1}, \\ \widetilde{\mathcal{D}_t v_1^{n+1}}|_{y=\eta_1^{n+1}} = (\widetilde{\mathcal{D}_t V_1^{n+1}}, \widetilde{\mathcal{D}_t B_1^{n+1}}), \quad \widetilde{\mathcal{D}_t v_1^{n+1}}|_{y=-1} = (\widetilde{\mathcal{D}_t^b V_{b,1}^{n+1}}, 0), \end{cases}$$

where

$$\begin{aligned} -h_\omega^{n+1} &= \nabla_{x,y} \times \widetilde{\mathcal{D}_t \omega^{n+1}} - \nabla_{x,y} v_2^{n+1} \cdot \nabla_{x,y} w^{n+1} + \Delta_{x,y} v_2^{n+1} \cdot \nabla_{x,y} v_1^{n+1} \\ &\quad + 2\partial_i v_2^{n+1} \cdot \nabla_{x,y} \partial_i v_1^{n+1}. \end{aligned}$$

We can deduce from Lemma 2.13, (11.2) and (11.3) that

$$\begin{aligned} \|\widetilde{h_\omega^{n+1}}\|_{H^{s-2}(\overline{\mathcal{S}})} &\leq P_{s-\frac{1}{2}, \eta_1^{n+1}}(\|\widetilde{\mathcal{D}_t \omega^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\widetilde{\omega^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})}^2 + \|\nabla_{x,z}(\widetilde{v^{n+1}}, \widetilde{v_1^{n+1}})\|_{H^{s-1}(\overline{\mathcal{S}})}^2) \\ &\leq P_{s-\frac{1}{2}, \eta_1^{n+1}}(\|\widetilde{\omega^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})}^2 + \|\nabla_{x,z}(\widetilde{v^{n+1}}, \widetilde{v_1^{n+1}})\|_{H^{s-1}(\overline{\mathcal{S}})}^2) \\ &\leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)). \end{aligned}$$

While, we know from Lemma 11.4 that

$$\|(\widetilde{\mathcal{D}_t V_1^{n+1}}, \widetilde{\mathcal{D}_t B_1^{n+1}}, \widetilde{\mathcal{D}_t^b V_{b,1}^{n+1}})\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_3(1 + \mathcal{E}^{n+1}(t)).$$

Then we apply Proposition 4.10 again to obtain

$$\|\nabla_{x,z} \widetilde{\mathcal{D}_t v_1^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)). \quad (11.4)$$

Similarly, we have

$$\|\nabla_{x,z} \widetilde{\mathcal{D}_t v^{n+1}}\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)) \quad (11.5)$$

by noting that

$$\|(\widetilde{\mathcal{D}_t V^{n+1}}, \widetilde{\mathcal{D}_t B^{n+1}}, \widetilde{\mathcal{D}_t^b V_b^{n+1}})\|_{H^{s-\frac{3}{2}}} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)).$$

Using (11.2)-(11.4), we can deduce that

Lemma 11.5. *It holds that*

$$\begin{aligned}\|\widetilde{\mathcal{D}_t \omega^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})} &\leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)), \\ \|\widetilde{\mathcal{D}_t^2 \omega^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})} &\leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)).\end{aligned}$$

For v_ω^{n+1} defined by

$$\begin{cases} -\Delta_{x,y} v_\omega^{n+1} = \nabla_{x,y} \times \omega^{n+1} & \text{in } \Omega_t^{n+1}, \\ v_\omega^{n+1}|_{y=\eta_1^{n+1}} = 0, \quad v_\omega^{n+1}|_{y=-1} = (V_{b,1}^{n+1}, 0), \end{cases}$$

we can deduce in a similar way that

$$\|\widetilde{v_\omega^{n+1}}\|_{H^s(\overline{\mathcal{S}})} \leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)), \quad (11.6)$$

$$\|\widetilde{\mathcal{D}_t v_\omega^{n+1}}\|_{H^s(\overline{\mathcal{S}})} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)), \quad (11.7)$$

$$\|\widetilde{\mathcal{D}_t v_\omega^{n+1}}\|_{X^{s-\frac{1}{2}}([-\frac{1}{2}, 0])} \leq \mathcal{P}(\mathcal{A}_2, \mathcal{E}_2^{n+1}(t)), \quad (11.8)$$

by using the fact that

$$\|\widetilde{\mathcal{D}_t^b V_{b,1}^{n+1}}\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_3, \quad \|\widetilde{\mathcal{D}_t^b V_{b,1}^{n+1}}\|_{H^{s-\frac{3}{2}}} \leq \mathcal{A}_2.$$

On the other hand, we know that

$$\begin{cases} \Delta_{x,y} \widetilde{\mathcal{D}_t^2 v_\omega^{n+1}} = \widetilde{\mathcal{D}_t} h_\omega^{n+1} + [\Delta_{x,y}, \widetilde{\mathcal{D}_t}] \widetilde{\mathcal{D}_t} v_\omega^{n+1} & \text{in } \Omega_t^{n+1}, \\ \widetilde{\mathcal{D}_t^2 v_\omega^{n+1}}|_{y=\eta_1^{n+1}} = 0, \quad \widetilde{\mathcal{D}_t^2 v_\omega^{n+1}}|_{y=-1} = ((\widetilde{\mathcal{D}_t^b})^2 V_{b,1}^{n+1}, 0), \end{cases}$$

where

$$\begin{aligned} h_\omega^{n+1} &= \nabla_{x,y} \times \widetilde{\mathcal{D}_t} \omega^{n+1} - \nabla_{x,y} v_2^{n+1} \cdot \nabla_{x,y} \omega^{n+1} - \nabla_{x,y} \omega^{n+1} \cdot \nabla_{x,y} v_\omega^{n+1} \\ &\quad + 2\partial_i v_2^{n+1} \cdot \nabla_{x,y} \partial_i v_\omega^{n+1}. \end{aligned}$$

Using (11.4)-(11.7) and Lemma 11.5, we can deduce from Lemma 2.13 that

$$\|[\Delta_{x,y}, \widetilde{\mathcal{D}_t}] \widetilde{\mathcal{D}_t} v_\omega^{n+1}\|_{H^{s-2}(\overline{\mathcal{S}})} + \|\widetilde{\mathcal{D}_t} h_\omega^{n+1}\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)).$$

And by Lemma 11.4, we have

$$\begin{aligned} \|(\widetilde{\mathcal{D}_t^b})^2 V_{b,1}^{n+1}\|_{H^{s-\frac{3}{2}}} &\leq \|\partial_t \widetilde{\mathcal{D}_t^b} V_{b,1}^{n+1}\|_{H^{s-\frac{3}{2}}} + \|V_b^{n+1}\|_{H^{s-\frac{3}{2}}} \|\nabla^2 P^n(\cdot, -1)\|_{H^{s-\frac{3}{2}}} \\ &\leq \mathcal{P}(\mathcal{A}_5, \mathcal{E}^{n+1}(t)). \end{aligned}$$

Then we can prove that

$$\|\nabla_{x,y} \widetilde{\mathcal{D}_t^2 v_\omega^{n+1}}\|_{L^2(\Omega_t^{n+1})} \leq \mathcal{P}(\mathcal{A}_5, \mathcal{E}^{n+1}(t)).$$

Thus, Proposition 4.9 ensures that

$$\|\nabla_{x,z} \widetilde{\mathcal{D}_t^2 v_\omega^{n+1}}\|_{X^{s-\frac{3}{2}}([-\frac{1}{2}, 0])} \leq \mathcal{P}(\mathcal{A}_5, \mathcal{E}^{n+1}(t)). \quad (11.9)$$

11.4. Estimate of the pressure. Recall that the pressure P^{n+1} satisfies

$$\begin{cases} -\Delta_{x,y} P^{n+1} = (\partial_i (v_1^j)^{n+1} \partial_j (v_1^i)^{n+1}) \circ \Phi_1^{n+1} \circ (\Phi^{n+1})^{-1} \triangleq F, \\ P^{n+1}|_{y=\eta^{n+1}} = 0, \quad (\partial_y P^{n+1})|_{y=-1} = -1. \end{cases}$$

Let $P_1^{n+1} = P^{n+1} + y$, $I_1 = [a, 0]$ for $a \in (-1, 0)$. First of all, we get by a similar proof of Lemma 7.1 that

$$\|\nabla_{x,y} P_1\|_{L^2(\tilde{\Omega}_t^{n+1})} \leq P_{s-\frac{1}{2}, \eta^{n+1}} \|\nabla_{x,z} \widetilde{v_1^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})}^2,$$

from which and Proposition 4.9, it follows that

$$\begin{aligned} \|\nabla_{x,z} \widetilde{P_1^{n+1}}\|_{X^{s-\frac{1}{2}}(I_1)} &\leq P_{s+\frac{1}{2}, \eta^{n+1}} (\|\nabla_{x,y} P_1^{n+1}\|_{L^2(\tilde{\Omega}_t^{n+1})} + P_{s-\frac{1}{2}, \eta_1^{n+1}} \|\nabla_{x,z} \widetilde{v_1^{n+1}}\|_{H^{s-1}(\overline{\mathcal{S}})}^2) \\ &\leq \mathcal{P}(\mathcal{A}_2, \mathcal{E}^{n+1}(t)). \end{aligned} \quad (11.10)$$

Similarly, we can deduce that

$$\|\nabla_{x,z} \widetilde{P_1^{n+1}}\|_{X^{s-\frac{3}{2}}(I_1)} \leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)). \quad (11.11)$$

These ensure that there exists $y_0 \in (-1, -1 + h_0]$ (in fact, one can take $y_0 > -1 + a$ with a depending on $\|\eta^{n+1}\|_{H^{s-\frac{1}{2}}}$) so that

$$\begin{aligned} \|P_1(\cdot, y_0)\|_{H^{s+\frac{1}{2}}} &\leq \mathcal{P}(\mathcal{A}_2, \mathcal{E}^{n+1}(t)), \\ \|P_1(\cdot, y_0)\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)). \end{aligned}$$

Then by the elliptic estimate in the flat strip, we obtain

$$\|\nabla_{x,y} P_1^{n+1}\|_{H^s(\mathbf{R}^d \times [-1, y_0])} \leq \mathcal{P}(\mathcal{A}_2, \mathcal{E}_2^{n+1}(t)), \quad (11.12)$$

$$\|\nabla_{x,y} P_1^{n+1}\|_{H^{s-1}(\mathbf{R}^d \times [-1, y_0])} \leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)). \quad (11.13)$$

Using Lemma 11.3, (11.3) and (11.4), we can deduce that

$$\|\widetilde{\mathcal{D}_t F}\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)).$$

It is easy to show that

$$\|\nabla_{x,y} \mathcal{D}_t P^{n+1}\|_{L^2(\tilde{\Omega}_t^{n+1})} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)).$$

Thus, we can deduce that

$$\|\nabla_{x,z} \widetilde{\mathcal{D}_t P^{n+1}}\|_{X^{s-\frac{3}{2}}(I_1)} + \|\nabla_{x,y} \mathcal{D}_t P^{n+1}(\cdot, -1)\|_{H^{s-\frac{3}{2}}} \leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)). \quad (11.14)$$

It follows from (11.10), (11.11) and (11.14) that

Lemma 11.6. *It holds that*

$$\begin{aligned} \|a^{n+1}\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)), \\ \|a^{n+1}\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{P}(\mathcal{A}_2, \mathcal{E}^{n+1}(t)), \\ \|\partial_t a^{n+1}\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{P}(\mathcal{A}_3, \mathcal{E}^{n+1}(t)). \end{aligned}$$

11.5. Estimates of the remainder of DN operator. In this subsection, we establish some estimates for the material derivatives of the remainder of DN operator, which will be used to estimate R_ω^n . For this purpose, we assume that η is a solution of the following equation

$$\partial_t \eta + V \cdot \nabla \eta = B,$$

where $(V, B) = v|_{y=\eta}$. Let $\overline{\mathcal{D}}_t \triangleq \partial_t + V \cdot \nabla$ and $\mathcal{D}_t \triangleq \partial_t + v \cdot \nabla_{x,y}$.

Now, we state the main result in this subsection:

Proposition 11.7. *Assume that $\eta, \overline{\mathcal{D}}_t \eta, \overline{\mathcal{D}}_t^2 \eta \in H^{s-\frac{1}{2}}(\mathbf{R}^d)$ for $s > \frac{d}{2} + 5$. Then it holds that*

$$\begin{aligned} \|\overline{\mathcal{D}}_t R(\eta) f\|_{H^{s-\frac{3}{2}}} &\leq P_{s-\frac{1}{2}, \eta, \overline{\mathcal{D}}_t \eta} \mathcal{V}_1(t) (\|f\|_{H^{s-\frac{1}{2}}} + \|\overline{\mathcal{D}}_t f\|_{H^{s-\frac{3}{2}}}), \\ \|\overline{\mathcal{D}}_t^2 R(\eta) f\|_{H^{s-\frac{3}{2}}} &\leq P_{s-\frac{1}{2}, \eta, \overline{\mathcal{D}}_t \eta, \overline{\mathcal{D}}_t^2 \eta} \mathcal{V}_2(t) (\|f\|_{H^{s-\frac{1}{2}}} + \|\overline{\mathcal{D}}_t f\|_{H^{s-\frac{1}{2}}} + \|\overline{\mathcal{D}}_t^2 f\|_{H^{s-\frac{3}{2}}}), \end{aligned}$$

where $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ are given by

$$\begin{aligned} \mathcal{V}_1(t) &\triangleq \mathcal{P}(\|V(t)\|_{H^{s-\frac{1}{2}}}, \|\nabla_{x,z} \tilde{v}(t)\|_{H^{s-1}(\overline{\mathcal{S}})}), \\ \mathcal{V}_2(t) &\triangleq \mathcal{P}(\|\nabla_{x,z}(\tilde{v}, \widetilde{\mathcal{D}_t v})(t)\|_{H^{s-1}(\overline{\mathcal{S}})}, \|(V, \overline{\mathcal{D}}_t V)\|_{H^{s-\frac{1}{2}}}) \end{aligned}$$

Let $\phi(t, x, y)$ be a solution of the elliptic equation

$$\begin{cases} \Delta_{x,y} \phi = 0 & \text{in } \Omega_t = \{(x, y) : -1 < y < \eta(t, x)\}, \\ \phi|_{y=\eta(t,x)} = f, & \phi|_{y=-1} = 0. \end{cases}$$

Proposition 4.10 ensures that

$$\|\tilde{\phi}\|_{H^s(\overline{\mathcal{S}})} \leq P_{s-\frac{1}{2}, \eta} \|f\|_{H^{s-\frac{1}{2}}}. \quad (11.15)$$

We next establish some estimates for $\mathcal{D}_t^k \phi$ for $k = 1, 2$.

Lemma 11.8. *It holds that*

$$\begin{aligned} \|\widetilde{\mathcal{D}_t \phi}\|_{H^s(\overline{\mathcal{S}})} &\leq P_{s-\frac{1}{2}, \eta} (\|\overline{\mathcal{D}}_t f\|_{H^{s-\frac{1}{2}}} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})} \|f\|_{H^{s-\frac{1}{2}}}), \\ \|\widetilde{\mathcal{D}_t^2 \phi}\|_{H^{s-1}(\overline{\mathcal{S}})} &\leq P_{s-\frac{1}{2}, \eta} \mathcal{V}(t) (\|\overline{\mathcal{D}}_t^2 f\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}}_t f\|_{H^{s-\frac{3}{2}}} + \|f\|_{H^{s-\frac{3}{2}}}), \\ \|(\partial_z - T_A) \widetilde{\mathcal{D}_t \phi}\|_{X^{s-\frac{3}{2}}([-\frac{1}{2}, 0])} &\leq P_{s-\frac{1}{2}, \eta} (\|\overline{\mathcal{D}}_t f\|_{H^{s-\frac{3}{2}}} + \|\nabla_{x,z} \tilde{v}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} \|f\|_{H^{s-\frac{1}{2}}}), \\ \|(\partial_z - T_A) \widetilde{\mathcal{D}_t^2 \phi}\|_{X^{s-\frac{3}{2}}([-\frac{1}{2}, 0])} &\leq P_{s-\frac{1}{2}, \eta} \mathcal{V}(t) (\|\overline{\mathcal{D}}_t^2 f\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}}_t f\|_{H^{s-\frac{1}{2}}} + \|f\|_{H^{s-\frac{1}{2}}}). \end{aligned}$$

Here $\mathcal{V}(t) \triangleq \mathcal{P}(\|\nabla_{x,z} \tilde{v}(t)\|_{H^{s-1}(\overline{\mathcal{S}})}, \|\nabla_{x,z} \widetilde{\mathcal{D}_t v}(t)\|_{H^{s-1}(\overline{\mathcal{S}})})$.

Proof. A direct calculations gives

$$\begin{cases} \Delta_{x,y} \mathcal{D}_t \phi = \Delta_{x,y} v \cdot \nabla_{x,y} \phi + 2 \nabla_{x,y} v \cdot \nabla_{x,y}^2 \phi \triangleq F, \\ \mathcal{D}_t \phi|_{y=\eta} = \overline{\mathcal{D}}_t f, \quad \mathcal{D}_t \phi|_{y=-1} = 0. \end{cases}$$

By Lemma 2.13, it is easy to show that

$$\|\tilde{F}\|_{H^{s-2}(\overline{\mathcal{S}})} \leq P_{s-\frac{1}{2}, \eta} \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})} \|\tilde{\phi}\|_{H^s(\overline{\mathcal{S}})}.$$

Then we deduce from Proposition 4.10 and (11.15) that

$$\|\widetilde{\mathcal{D}_t \phi}\|_{H^s(\overline{\mathcal{S}})} \leq P_{s-\frac{1}{2}, \eta} (\|\overline{\mathcal{D}}_t f\|_{H^{s-\frac{1}{2}}} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})} \|f\|_{H^{s-\frac{1}{2}}}). \quad (11.16)$$

Similar, we have

$$\|\widetilde{\mathcal{D}_t \phi}\|_{H^{s-1}(\overline{\mathcal{S}})} \leq P_{s-\frac{1}{2},\eta} (\|\overline{\mathcal{D}_t f}\|_{H^{s-\frac{3}{2}}} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-2}(\overline{\mathcal{S}})} \|f\|_{H^{s-\frac{3}{2}}}).$$

Then a similar argument of Proposition 5.2 implies the third inequality.

By Lemma 2.13 again, we have

$$\begin{aligned} \|\widetilde{\mathcal{D}_t F}\|_{H^{s-3}(\overline{\mathcal{S}})} &\leq P_{s-\frac{1}{2},\eta} \|\nabla_{x,z} \tilde{v}\|_{H^{s-2}(\overline{\mathcal{S}})}^2 \|\widetilde{\mathcal{D}_t \phi}\|_{H^{s-1}(\overline{\mathcal{S}})} \\ &\quad + P_{s-\frac{1}{2},\eta} (\|\nabla_{x,z} \widetilde{\mathcal{D}_t v}\|_{H^{s-2}(\overline{\mathcal{S}})} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-2}(\overline{\mathcal{S}})}^2) \|\tilde{\phi}\|_{H^{s-1}(\overline{\mathcal{S}})}. \end{aligned}$$

This implies the second inequality. We also have

$$\begin{aligned} \|\widetilde{\mathcal{D}_t F}\|_{H^{s-2}(\overline{\mathcal{S}})} &\leq P_{s-\frac{1}{2},\eta} \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})}^2 \|\widetilde{\mathcal{D}_t \phi}\|_{H^s(\overline{\mathcal{S}})} \\ &\quad + P_{s-\frac{1}{2},\eta} (\|\nabla_{x,z} \widetilde{\mathcal{D}_t v}\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})}^2) \|\tilde{\phi}\|_{H^s(\overline{\mathcal{S}})} \\ &\leq P_{s-\frac{1}{2},\eta} \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})}^2 (\|\overline{\mathcal{D}_t f}\|_{H^{s-\frac{1}{2}}} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})} \|f\|_{H^{s-\frac{1}{2}}}) \\ &\quad + P_{s-\frac{1}{2},\eta} (\|\nabla_{x,z} \widetilde{\mathcal{D}_t v}\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})}^2) \|f\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

Then a similar argument of Proposition 5.2 implies the last inequality. \square

Lemma 11.9. *It holds that*

$$\begin{aligned} \|\overline{\mathcal{D}_t \tilde{\phi}}|_{z=0}\|_{H^{s-\frac{3}{2}}} &\leq P_{s-\frac{1}{2},\eta} (\|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})} \|f\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}_t f}\|_{H^{s-\frac{3}{2}}}), \\ \|\overline{\mathcal{D}_t^2 \tilde{\phi}}|_{z=0}\|_{H^{s-\frac{3}{2}}} &\leq P_{s-\frac{1}{2},\eta} \mathcal{V}(t) (\|f\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}_t f}\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}_t^2 f}\|_{H^{s-\frac{3}{2}}}). \end{aligned}$$

In addition, we have

$$\begin{aligned} \|\overline{\mathcal{D}_t \nabla_{x,z} \tilde{\phi}}|_{z=0}\|_{H^{s-\frac{5}{2}}} &\leq P_{s-\frac{1}{2},\eta, \overline{\mathcal{D}_t \eta}} (\|\nabla_{x,z} \tilde{v}\|_{H^{s-1}(\overline{\mathcal{S}})} \|f\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}_t f}\|_{H^{s-\frac{3}{2}}}), \\ \|\overline{\mathcal{D}_t^2 \nabla_{x,z} \tilde{\phi}}|_{z=0}\|_{H^{s-\frac{5}{2}}} &\leq P_{s-\frac{1}{2},\eta, \overline{\mathcal{D}_t \eta}, \overline{\mathcal{D}_t^2 \eta}} \mathcal{V}(t) (\|f\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}_t f}\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}_t^2 f}\|_{H^{s-\frac{3}{2}}}). \end{aligned}$$

Proof. The first two inequalities follows from Lemma 11.8 and the fact that

$$\overline{\mathcal{D}_t^k \tilde{\phi}}|_{z=0} = \widetilde{\mathcal{D}_t^k \phi}|_{z=0}.$$

The last two inequalities can be deduced from Lemma 2.11, Lemma 11.8 and the formulas

$$\begin{aligned} \overline{\mathcal{D}_t \nabla \tilde{\phi}}|_{z=0} &= \widetilde{\mathcal{D}_t \nabla \phi}|_{z=0} + \widetilde{\mathcal{D}_t \partial_y \phi}|_{z=0} \nabla \eta + \widetilde{\partial_y \phi}|_{z=0} \overline{\mathcal{D}_t \nabla \eta}, \\ \overline{\mathcal{D}_t \partial_z \tilde{\phi}}|_{z=0} &= \widetilde{\mathcal{D}_t \partial_y \phi}|_{z=0} (1 + \eta + \delta |D| \eta) + \widetilde{\partial_y \phi}|_{z=0} \overline{\mathcal{D}_t (\eta + \delta |D| \eta)}. \end{aligned}$$

We omit the details. \square

Proof of Proposition 11.7. We write

$$\begin{aligned} \overline{\mathcal{D}_t R_1(\eta) f} &= [\overline{\mathcal{D}_t}, T_{\zeta_1} T_A - T_{\zeta_1 A}] \tilde{\phi}|_{z=0} + (T_{\zeta_1} T_A - T_{\zeta_1 A}) \widetilde{\mathcal{D}_t \phi}|_{z=0} \\ &= (T_{\partial_t \zeta_1} T_A + T_{\zeta_1} T_{\partial_t A} - T_{\partial_t (\zeta_1 A)}) \tilde{\phi}|_{z=0} + [V \cdot \nabla, T_{\zeta_1} T_A - T_{\zeta_1 A}] \tilde{\phi}|_{z=0} \\ &\quad + (T_{\zeta_1} T_A - T_{\zeta_1 A}) \widetilde{\mathcal{D}_t \phi}|_{z=0}. \end{aligned}$$

Using Proposition 2.6 and Lemma 11.9, and the following trick

$$\|\partial_t g\|_{H^{s-\frac{5}{2}}} \leq \|\overline{\mathcal{D}_t g}\|_{H^{s-\frac{5}{2}}} + \|V\|_{H^{s-\frac{5}{2}}} \|g\|_{H^{s-\frac{3}{2}}},$$

we can deduce that

$$\|\overline{\mathcal{D}_t R_1(\eta) f}\|_{H^{s-\frac{3}{2}}} \leq P_{s-\frac{1}{2},\eta, \overline{\mathcal{D}_t \eta}} \mathcal{V}(t) (\|f\|_{H^{s-\frac{3}{2}}} + \|\overline{\mathcal{D}_t f}\|_{H^{s-\frac{3}{2}}}).$$

Similarly, we have

$$\|\overline{D}_t R_2(\eta) f\|_{H^{s-\frac{3}{2}}} \leq P_{s-\frac{1}{2}, \eta, \overline{D}_t \eta} \mathcal{V}(t) (\|f\|_{H^{s-\frac{1}{2}}} + \|\overline{D}_t f\|_{H^{s-\frac{3}{2}}}).$$

The same estimates holds for $\overline{D}_t R_3(\eta)$ by using the following type estimate

$$\begin{aligned} \|\overline{D}_t R(f, g)\|_{H^{s-\frac{3}{2}}} &\leq \|R(\partial_t f, g) + R(f, \partial_t g) + V \cdot \nabla R(f, g)\|_{H^{s-\frac{3}{2}}} \\ &\leq C \|\partial_t f\|_{H^{s-\frac{5}{2}}} \|g\|_{H^{s-\frac{3}{2}}} + C \|\partial_t g\|_{H^{s-\frac{5}{2}}} \|f\|_{H^{s-\frac{3}{2}}} + C \|V\|_{H^{s-\frac{3}{2}}} \|f\|_{H^{s-\frac{3}{2}}} \|g\|_{H^{s-\frac{3}{2}}}. \end{aligned}$$

The estimates for $\overline{D}_t^2 R(\eta) f$ can be similarly proved by a very tedious computation with the help of Lemma 11.8 and Lemma 11.9. We omit the details. \square

Applying Proposition 11.7 with $v = v_1^n$ and $(V, B) = (V_1^n, B_1^n)$, we deduce that

Lemma 11.10. *It holds that*

$$\begin{aligned} \|D_t^u R(\eta_1^n)(V_1^n, B_1^n)\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{P}(E_1, E_2, E_4), \\ \|(D_t^u)^2 R(\eta_1^n)(V_1^n, B_1^n)\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_6. \end{aligned}$$

11.6. Energy estimates.

Proposition 11.11. *It holds that*

$$\|\tilde{\omega}^{n+1}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} \leq \|\tilde{\omega}_0\|_{H^{s-1}(\overline{\mathcal{S}})} + \int_0^t \mathcal{P}(\mathcal{A}_1, \mathcal{E}^{n+1}(t')) dt'.$$

Proof. Recall that the vorticity ω^{n+1} satisfies

$$\begin{cases} \partial_t \omega^{n+1} + ((v^{n+1})^h \cdot \nabla + (v_1^{n+1})^{d+1} \partial_y) \omega^{n+1} = \omega^{n+1} \cdot \nabla_{x,y} v_1^{n+1}, \\ \omega^{n+1}|_{t=0} = \omega_0. \end{cases}$$

In terms of the (x, z) variable, we have

$$\partial_t \tilde{\omega}^{n+1} + \overline{v}^{n+1} \cdot \nabla_{x,z} \tilde{\omega}^{n+1} = F^{n+1}, \quad (11.17)$$

where

$$\begin{aligned} \overline{v}^{n+1} &= ((\tilde{v}^{n+1})^h, \frac{1}{\partial_z \rho_{\delta, \eta_1^{n+1}}} (\tilde{v}_1^{d+1} - \partial_t \rho_{\delta, \eta_1^{n+1}} - (\tilde{v}^{n+1})^h \cdot \nabla \rho_{\delta, \eta_1^{n+1}})), \\ F^{n+1} &= (\tilde{\omega}^{n+1})^h \cdot (\nabla \tilde{v}_1^{n+1} - \frac{\nabla \rho_{\delta, \eta_1^{n+1}}}{\partial_z \rho_{\delta, \eta_1^{n+1}}} \partial_z \tilde{v}_1) + (\tilde{\omega}^{n+1})^{d+1} \frac{\partial_z \tilde{v}_1}{\partial_z \rho_{\delta, \eta_1^{n+1}}}. \end{aligned}$$

By Lemma 2.13, it is easy to see that

$$\|F^{n+1}\|_{H^{s-1}(\overline{\mathcal{S}})} \leq \mathcal{P}_{s-\frac{1}{2}, \eta_1^{n+1}} \|\nabla_{x,z} (\widetilde{v^{n+1}}, \widetilde{v_1^{n+1}})\|_{H^{s-1}(\overline{\mathcal{S}})} \|\tilde{\omega}^{n+1}\|_{H^{s-1}(\overline{\mathcal{S}})}.$$

By Lemma 9.6, we have

$$\|\nabla_{x,z} \overline{v}^{n+1}\|_{H^{s-1}(\overline{\mathcal{S}})} \leq \mathcal{P}_{s-\frac{1}{2}, \eta_1^{n+1}} \|\nabla_{x,z} (\widetilde{v^{n+1}}, \widetilde{v_1^{n+1}})\|_{H^{s-1}(\overline{\mathcal{S}})}.$$

Then by a similar proof of Proposition 9.5, we deduce

$$\frac{d}{dt} \|\tilde{\omega}^{n+1}\|_{H^{s-1}(\overline{\mathcal{S}})} \leq \mathcal{P}_{s-\frac{1}{2}, \eta_1^{n+1}} \|\nabla_{x,z} (\widetilde{v^{n+1}}, \widetilde{v_1^{n+1}})\|_{H^{s-1}(\overline{\mathcal{S}})}. \quad (11.18)$$

This together with (11.2) and (11.3) gives the proposition. \square

Proposition 11.12. *It holds that*

$$\begin{aligned} & \| (V^{n+1}, B^{n+1}, V_b^{n+1})(t) \|_{H^{s-\frac{1}{2}}} + \|\zeta^{n+1}(t)\|_{H^{s-\frac{3}{2}}} \\ & \leq \left(\| (V_0, B_0, V_{b,0}) \|_{H^{s-\frac{1}{2}}} + \mathcal{A}_3 \int_0^t (1 + \mathcal{E}^{n+1}(t')) dt' \right) e^{t\mathcal{A}_1}. \end{aligned}$$

Proof. It follows from Lemma 11.4 that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|V^{n+1}\|_{H^{s-\frac{1}{2}}}^2 & \leq \mathcal{A}_3 (1 + \mathcal{E}^{n+1}(t)) \|V^{n+1}\|_{H^{s-\frac{1}{2}}} \\ & \quad + \langle \langle D \rangle^{s-\frac{1}{2}} T_{V^n} \cdot \nabla V^{n+1}, \langle D \rangle^{s-\frac{1}{2}} V^{n+1} \rangle. \end{aligned}$$

We write

$$\begin{aligned} & \langle \langle D \rangle^{s-\frac{1}{2}} T_{V^n} \cdot \nabla V^{n+1}, \langle D \rangle^{s-\frac{1}{2}} V^{n+1} \rangle \\ & = \langle [\langle D \rangle^{s-\frac{1}{2}}, T_{V^n} \cdot \nabla] V^{n+1}, \langle D \rangle^{s-\frac{1}{2}} V^{n+1} \rangle - \frac{1}{2} \langle T_{\nabla \cdot V^n} \langle D \rangle^{s-\frac{1}{2}} V^{n+1}, \langle D \rangle^{s-\frac{1}{2}} V^{n+1} \rangle, \end{aligned}$$

by Lemma 2.19, which is bounded by

$$C \|V^n\|_{W^{1,\infty}} \|V^{n+1}\|_{H^{s-\frac{1}{2}}}^2 \leq C \|V^n\|_{H^{s-\frac{1}{2}}} \|V^{n+1}\|_{H^{s-\frac{1}{2}}}^2.$$

This shows that

$$\frac{d}{dt} \|V^{n+1}\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_3 (1 + \mathcal{E}^{n+1}(t)) + C \|V^n\|_{H^{s-\frac{1}{2}}} \|V^{n+1}\|_{H^{s-\frac{1}{2}}}.$$

Then Gronwall's inequality ensures that

$$\|V^{n+1}(t)\|_{H^{s-\frac{1}{2}}} \leq \left(\|V_0\|_{H^{s-\frac{1}{2}}} + \mathcal{A}_3 \int_0^t (1 + \mathcal{E}^{n+1}(t')) dt' \right) e^{t\mathcal{A}_1}.$$

The estimates for the other terms are similar. \square

Similarly, we can prove by Lemma 11.4 and Lemma 11.3 that

Proposition 11.13. *It holds that*

$$\begin{aligned} & \| (V_1^{n+1}, B_1^{n+1}, V_{b,1}^{n+1})(t) \|_{H^{s-\frac{1}{2}}} + \|\eta_1^{n+1}(t)\|_{H^{s-\frac{1}{2}}} \\ & \leq \left(\| (V_0, B_0, V_{b,0}, \eta_0) \|_{H^{s-\frac{1}{2}}} + \mathcal{A}_3 \int_0^t (1 + \mathcal{E}^{n+1}(t')) dt' \right) e^{C \int_0^t \mathcal{E}_2^{n+1}(t') dt'}. \end{aligned}$$

Following the proof of Proposition 9.4, we can deduce that

Proposition 11.14. *It holds that*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|D_t \zeta^{n+1}\|_{H^{s-1}}^2 + \|T_{\sqrt{a^n \lambda^n}} \zeta^{n+1}\|_{H^{s-1}}^2) & \leq \mathcal{A}_4 (\|D_t \zeta_1^{n+1}\|_{H^{s-1}}^2 + \|\zeta\|_{H^{s-\frac{1}{2}}}^2) \\ & \quad + \langle (f_1^n + f_2^n + D_t f_3^n + D_t f_4^n + D_t f_\omega^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle. \end{aligned}$$

11.7. Nonlinear estimates. Recall that

$$f_1^n = D_t ((T_{V_1^n} - V_1^n) \cdot \nabla \zeta^n + [T_{\zeta^n}, T_{\lambda^n}] B_1^n) + (T_{a^n \lambda^n} - T_{\lambda^n} T_{a^n}) \zeta^{n+1}.$$

Lemma 11.15. *It holds that*

$$\|f_1^n\|_{H^{s-1}} \leq \mathcal{A}_5.$$

Proof. We write

$$D_t(T_{V_1^n} - V_1^n) \cdot \nabla \zeta^n = -[D_t, T_{\nabla \zeta^n}] \cdot V_1^n + T_{\nabla \zeta^n} \cdot D_t V_1^n - D_t R(\nabla \zeta^n, V_1^n).$$

It follows from Proposition 2.21 that

$$\|[D_t, T_{\nabla \zeta^n}] \cdot V_1^n\|_{H^{s-1}} \leq C(E_1 + E_4)\|V_1^n\|_{H^{s-1}} \leq \mathcal{A}_4.$$

And by Lemma 2.10, we get

$$\|T_{\nabla \zeta^n} \cdot D_t V_1^n\|_{H^{s-1}} \leq CE_1\|D_t V_1^n\|_{H^{s-1}} \leq \mathcal{A}_5.$$

By Lemma 2.10 again, we get

$$\begin{aligned} \|D_t R(\nabla \zeta^n, V_1^n)\|_{H^{s-1}} &\leq \|R(\nabla \partial_t \zeta^n, V_1^n)\|_{H^{s-1}} + \|R(\nabla \zeta^n, \partial_t V_1^n)\|_{H^{s-1}} \\ &\quad + \|T_{V_1^n} \cdot \nabla R(\nabla \zeta^n, V_1^n)\|_{H^s} \leq \mathcal{A}_4. \end{aligned}$$

We write

$$\begin{aligned} D_t[T_{\zeta^n}, T_{\lambda^n}]B_1^n &= [T_{\partial_t \zeta^n}, T_{\lambda^n}]B_1^n + [T_{\zeta^n}, T_{\partial_t \lambda^n}]B_1^n + [T_{\zeta^n}, T_{\lambda^n}]D_t B_1^n \\ &\quad + [T_{V_1^n} \cdot \nabla, [T_{\zeta^n}, T_{\lambda^n}]]B_1^n, \end{aligned}$$

which along with Proposition 2.6 and Corollary 2.9 gives

$$\|D_t[T_{\zeta^n}, T_{\lambda^n}]B_1^n\|_{H^{s-1}} \leq \mathcal{A}_4.$$

The proof is finished. \square

Recall that

$$f_2^n = [D_t, T_{\lambda^n}]U^{n+1} + T_{\lambda^n}(h_1^n + [D_t, T_{\zeta^n}]B^{n+1}).$$

Lemma 11.16. *It holds that*

$$\begin{aligned} \|f_2^n\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{P}(E_1, E_2, E_4)(1 + \mathcal{E}_2^{n+1}(t)), \\ \|D_t f_2^n\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_5(1 + \mathcal{E}^{n+1}(t)). \end{aligned}$$

Proof. The first inequality is obvious. We turn to the second inequality. It follows from Proposition 2.23 and Lemma 11.2 that

$$\begin{aligned} \|D_t[D_t, T_{\lambda^n}]U^{n+1}\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_5(\|U^{n+1}\|_{H^{s-\frac{1}{2}}} + \|D_t U^{n+1}\|_{H^{s-\frac{1}{2}}}) \\ &\leq \mathcal{A}_5(1 + \mathcal{E}^{n+1}(t)). \end{aligned}$$

We write

$$\begin{aligned} D_t T_{\lambda^n}(h_1^n + [D_t, T_{\zeta^n}]B^{n+1}) &= [D_t, T_{\lambda^n}](h_1^n + [D_t, T_{\zeta^n}]B^{n+1}) \\ &\quad + T_{\lambda^n}(D_t h_1^n + D_t[D_t, T_{\zeta^n}]B^{n+1}), \end{aligned}$$

which along with Proposition 2.21 and Proposition 2.6 gives

$$\begin{aligned} \|D_t T_{\lambda^n}(h_1^n + [D_t, T_{\zeta^n}]B^{n+1})\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{A}_5(\|h_1^n\|_{H^{s-\frac{1}{2}}} + \|B^{n+1}\|_{H^{s-\frac{1}{2}}}) \\ &\quad + \|D_t h_1^n\|_{H^{s-\frac{1}{2}}} + \|D_t B^{n+1}\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

Similar to Lemma 11.15, we can prove that

$$\|D_t h_1^n\|_{H^{s-\frac{3}{2}}} \leq \mathcal{A}_4.$$

This together with Lemma 11.4 gives

$$\|D_t T_{\lambda^n}(h_1^n + [D_t, T_{\zeta^n}]B^{n+1})\|_{H^{s-\frac{3}{2}}} \leq \mathcal{A}_5(1 + \mathcal{E}^{n+1}(t)).$$

Putting the above estimates together gives the second inequality. \square

Recall that

$$\begin{aligned} f_3^n &= (\zeta^n - T_{\zeta^n})T_{\lambda^n}B^n + (\zeta^n - T_{\zeta^n})R(\eta_1^n)B_1^n, \\ f_4^n &= R(\eta_1^n)V_1^n + T_{\zeta^n}R(\eta_1^n)B_1^n. \end{aligned}$$

Then we can deduce from Proposition 5.2 and Lemma 11.10 that

Lemma 11.17. *It holds that*

$$\begin{aligned} \|f_3^n\|_{H^{s-1}} &\leq \mathcal{A}_1 \|\eta^n\|_{H^s}, \quad \|f_3^n\|_{H^{s-\frac{1}{2}}} \leq \mathcal{A}_3, \\ \|D_t^u f_4^n\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{P}(E_1, E_2, E_4), \quad \|(D_t^u)^2 f_4^n\|_{H^{s-\frac{3}{2}}} \leq \mathcal{A}_6. \end{aligned}$$

Finally, we have

Lemma 11.18. *It holds that*

$$\|D_t^u f_\omega^n\|_{H^{s-\frac{3}{2}}} \leq \mathcal{P}(E_1, E_2, E_4), \quad \|(D_t^u)^2 f_\omega^n\|_{H^{s-\frac{3}{2}}} \leq \mathcal{A}_6.$$

11.8. Completion of the uniform estimate. Let

$$\mathcal{E}_0 \triangleq \|\widetilde{\omega}_0\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\eta_0\|_{H^{s+\frac{1}{2}}} + \|\widetilde{v}_0\|_{H^{s+\frac{1}{2}}(\overline{\mathcal{S}})} + \|D_t \zeta(0)\|_{H^{s-1}} + \|T_{\sqrt{a_0 \lambda_0}} \zeta_0\|_{H^{s-1}},$$

which is bounded by $\mathcal{P}(\|\eta_0\|_{H^{s+\frac{1}{2}}})(\|v_0\|_{H^{s+\frac{1}{2}}(\Omega_0)} + \|\eta_0\|_{H^{s+\frac{1}{2}}})$.

Let us first assume that there exists a maximal time $T_n \in (0, T]$ so that the solution satisfies

$$\mathcal{E}^k(t) \leq \mathcal{P}_0, \quad \mathcal{E}_2^k(t) \leq \mathcal{P}_1. \quad (11.19)$$

for $k = 1, \dots, n+1$ and $t \in [0, T_n)$ and some $\mathcal{P}_0, \mathcal{P}_1$ determined later. Under this assumption, we will verify (H1) – (H8) for the solution in the $(n+1)$ -th iteration. Moreover, we will show that

$$\mathcal{E}^{n+1}(t) \leq \frac{\mathcal{P}_0}{2}, \quad \mathcal{E}_2^{n+1}(t) \leq \frac{\mathcal{P}_1}{2}. \quad (11.20)$$

which implies that $T_n = T$ by a continuous argument.

Obviously, we can take $E_1 = \mathcal{P}_1$. If $(\eta^{n+1}, \eta_1^{n+1})$ satisfies (H8), then we deduce from (11.3), (11.6), Lemma 11.6 and (11.12) that

$$\begin{aligned} \|\nabla_{x,z} \widetilde{v}_1^n(t)\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\widetilde{v}_\omega^{n+1}(t)\|_{H^s(\overline{\mathcal{S}})} + \|a^{n+1}(t)\|_{H^{s-\frac{3}{2}}} \\ + \|\nabla P^{n+1}(t, \cdot, -1)\|_{H^{s-\frac{3}{2}}} \leq \mathcal{P}(\mathcal{E}_2^{n+1}(t)) \leq C(\mathcal{P}_1) \triangleq E_2. \end{aligned}$$

This in turn implies that we can take T_1 small enough depending on h_0, \mathcal{P}_1, E_2 so that for $t \in [0, \min(T_n, T_1))$,

$$\eta^{n+1}(t, x) + 1 \geq \frac{h_0}{2}, \quad \eta_1^{n+1}(t, x) + 1 \geq \frac{h_0}{2}.$$

By Lemma 11.6, Lemma 11.1 and (11.12), we have

$$\begin{aligned} \|a^{n+1}(t)\|_{H^{s-\frac{1}{2}}} + \|\nabla P^{n+1}(t, \cdot, -1)\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{P}(\mathcal{A}_2, \mathcal{E}^{n+1}(t)) \leq C(\mathcal{P}_0, \mathcal{P}_1) \triangleq E_3^1, \\ \|\eta^{n+1}(t)\|_{H^{s+\frac{1}{2}}} &\leq \mathcal{A}_2 \mathcal{E}^{n+1}(t) \leq \mathcal{A}_2 \mathcal{P}_0 \triangleq E_3^2. \end{aligned}$$

Then we deduce from Proposition 11.11, Proposition 11.12 and Proposition 11.13 that

$$\mathcal{E}_2^{n+1}(t) \leq \left(\mathcal{E}_2^{n+1}(0) + \mathcal{A}_3 \int_0^t \mathcal{P}(\mathcal{E}^{n+1}(t')) dt' \right) e^{\mathcal{A}_1 t + C \int_0^t \mathcal{E}^{n+1}(t') dt'}$$

$$\leq C_1(\mathcal{E}_0 + tC(\mathcal{P}_0, \mathcal{P}_1))e^{tC(\mathcal{P}_0, \mathcal{P}_1)},$$

which ensures that we can take $\mathcal{P}_1 = 2C_1\mathcal{E}_0$ and T_2 small enough depending on $\mathcal{E}_0, \mathcal{P}_0$ so that for $t \in [0, \min(T_n, T_1, T_2))$,

$$\mathcal{E}_2^{n+1}(t) \leq \frac{\mathcal{P}_1}{2}.$$

By Lemma 11.4, Lemma 11.3, Lemma 11.5 and (11.8), we have

$$\begin{aligned} & \|D_t(V_1^{n+1}, B_1^{n+1})(t)\|_{H^{s-\frac{3}{2}}} + \|\partial_t \eta^{n+1}(t)\|_{H^{s-\frac{1}{2}}} + \|\widetilde{D}_t^u \eta_1^{n+1}(t)\|_{H^{s-\frac{1}{2}}} \\ & + \|\widetilde{\mathcal{D}_t \omega^{n+1}}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\widetilde{\mathcal{D}_t v_\omega^{n+1}}(t)\|_{X^{s-\frac{1}{2}}([-\frac{1}{2}, 0])} \\ & \leq \mathcal{P}(\mathcal{A}_2, \mathcal{E}_2^{n+1}(t)) \leq C(\mathcal{P}_1) \triangleq E_4. \end{aligned}$$

By Lemma 11.4, Lemma 11.3, Lemma 11.6, (11.4) and (11.14), we obtain

$$\begin{aligned} & \|(D_t(V_1^{n+1}, B_1^{n+1}), D_t^b V_{b,1}^{n+1})(t)\|_{H^{s-\frac{1}{2}}} + \|\partial_t^2 \eta^{n+1}(t)\|_{H^{s-\frac{5}{2}}} + \|\partial_t a^{n+1}(t)\|_{H^{s-\frac{3}{2}}} \\ & + \|\widetilde{\mathcal{D}_t^2 \omega^n}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\nabla_{x,z} \widetilde{\mathcal{D}_t v_1^{n+1}}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\nabla \partial_t P^{n+1}(t, \cdot, -1)\|_{H^{s-\frac{3}{2}}} \\ & + \|(\widetilde{D}_t^u)^2 \eta_1^{n+1}(t)\|_{H^{s-\frac{1}{2}}} \leq \mathcal{P}(\mathcal{A}_4, \mathcal{E}^{n+1}(t)) \leq C(\mathcal{P}_0, \mathcal{P}_1) \triangleq E_5. \end{aligned}$$

While, (H5) and the initial conditions (1.7) ensure that there exists T_3 small enough depending on E_5 so that (H7) holds for $t \in [0, \min(T_n, T_1, T_2, T_3))$.

By Lemma 11.4, Lemma 11.5 and (11.9), we have

$$\begin{aligned} & \|\partial_t(D_t(V_1^{n+1}, B_1^{n+1}), D_t^b V_{b,1}^{n+1})(t)\|_{H^{s-\frac{3}{2}}} + \|\widetilde{\mathcal{D}_t^2 v_\omega^{n+1}}(t)\|_{H^{s-1}(\overline{\mathcal{S}})} \\ & \leq \mathcal{P}(\mathcal{A}_5, \mathcal{E}^{n+1}(t)) \leq \mathcal{P}(\mathcal{A}_5, \mathcal{P}_0) \triangleq E_6. \end{aligned}$$

To complete the proof, it remains to prove the first inequality of (11.20). We know from Proposition 11.14 that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|D_t \zeta^{n+1}\|_{H^{s-1}}^2 + \|T_{\sqrt{a^n \lambda^n}} \zeta^{n+1}\|_{H^{s-1}}^2) \leq \mathcal{A}_4 (\|D_t \zeta^{n+1}\|_{H^{s-1}}^2 + \|\zeta\|_{H^{s-\frac{1}{2}}}^2) \\ & + \langle (f_1^n + f_2^n + D_t f_3^n + D_t f_4^n + f_\omega^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle. \end{aligned}$$

By Lemma 11.15, we have

$$\langle (f_1^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle \leq \mathcal{A}_5 \|D_t \zeta^{n+1}\|_{H^{s-1}}.$$

We write

$$\begin{aligned} \langle (f_2^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle & = \langle (f_2^n)_{s-1}, [\langle D \rangle^s, D_t] \zeta^{n+1} \rangle + \langle (D_t)^* (f_2^n)_{s-1}, (\zeta^{n+1})_{s-1} \rangle \\ & + \frac{d}{dt} \langle (f_2^n)_{s-1}, (\zeta^{n+1})_{s-1} \rangle, \end{aligned}$$

from which, Lemma 2.19, Lemma 11.16 and Lemma 11.1, we deduce that

$$\begin{aligned} \langle (f_2^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle & = \frac{d}{dt} \langle (f_2^n)_{s-1}, (\zeta^{n+1})_{s-1} \rangle \\ & + \mathcal{A}_2 (\|f_2^n\|_{H^{s-\frac{3}{2}}} + \|D_t f_2^n\|_{H^{s-\frac{3}{2}}}) \|\zeta^{n+1}\|_{H^{s-\frac{1}{2}}} \\ & = \frac{d}{dt} \langle (f_2^n)_{s-1}, (\zeta^{n+1})_{s-1} \rangle + \mathcal{A}_5 \mathcal{E}^{n+1}(t). \end{aligned}$$

In a similar way, we deduce from Lemma 11.18 and Lemma 11.17 that

$$\begin{aligned}\langle (D_t f_\omega^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle &= \frac{d}{dt} \langle (D_t^u f_\omega^n)_{s-1}, (\zeta^{n+1})_{s-1} \rangle + \mathcal{A}_6 \mathcal{E}^{n+1}(t), \\ \langle (D_t f_4^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle &= \frac{d}{dt} \langle (D_t^u f_4^n)_{s-1}, (\zeta^{n+1})_{s-1} \rangle + \mathcal{A}_6 \mathcal{E}^{n+1}(t).\end{aligned}$$

We write

$$\begin{aligned}\langle (D_t f_3^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle &= \langle [\langle D \rangle^{s-1}, D_t] f_3^n, (D_t \zeta^{n+1})_{s-1} \rangle \\ &\quad + \langle (f_3^n)_{s-1}, (D_t)^*(D_t \zeta^{n+1})_{s-1} \rangle + \frac{d}{dt} \langle (f_3^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle,\end{aligned}$$

which along with Lemma 2.19 and Lemma 11.17 gives

$$\begin{aligned}\langle (D_t f_3^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle &= \frac{d}{dt} \langle (f_3^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle \\ &\quad + \mathcal{A}_2 \|f_3^n\|_{H^{s-\frac{1}{2}}} (\|D_t \zeta^{n+1}\|_{H^{s-1}} + \|D_t^2 \zeta^{n+1}\|_{H^{s-\frac{3}{2}}}) \\ &= \frac{d}{dt} \langle (f_3^n)_{s-1}, (D_t \zeta^{n+1})_{s-1} \rangle + \mathcal{A}_5 \mathcal{E}^{n+1}(t) + \mathcal{A}_5.\end{aligned}$$

Putting the above estimates together, we obtain

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} (\|D_t \zeta^{n+1}\|_{H^{s-1}}^2 + \|T_{\sqrt{a^n \lambda^n}} \zeta^{n+1}\|_{H^{s-1}}^2) &\leq \mathcal{A}_5 \mathcal{E}^{n+1}(t)^2 + \mathcal{A}_6 \\ &\quad + \frac{d}{dt} \langle (f_2^n)_{s-1} + (f_3^n)_{s-1} + (D_t^u f_4^n)_{s-1} + (D_t^u f_\omega^n)_{s-1}, (\zeta^{n+1})_{s-1} \rangle.\end{aligned}$$

Integrating in t and using Lemma 11.16-Lemma 11.18, we deduce that

$$\begin{aligned}\|D_t \zeta^{n+1}\|_{H^{s-1}}^2 + \|T_{\sqrt{a^n \lambda^n}} \zeta^{n+1}\|_{H^{s-1}}^2 &\leq \mathcal{P}(\mathcal{E}_0) + \mathcal{A}_6 \int_0^t (1 + \mathcal{E}^{n+1}(t')^2) dt' \\ &\quad + \mathcal{P}(E_1, E_2, E_4) + \mathcal{A}_1 \|\eta^n\|_{H^s}^2 + \epsilon \|\zeta^{n+1}\|_{H^{s-\frac{1}{2}}}^2 \\ &\leq \mathcal{P}(\mathcal{E}_0) + \mathcal{P}(E_1, E_2, E_4) + \mathcal{A}_6 (1 + \mathcal{P}_0)^2 t + \frac{1}{64} \mathcal{P}_0^2 + \frac{1}{4} \|T_{\sqrt{a^n \lambda^n}} \zeta^{n+1}\|_{H^{s-1}}^2.\end{aligned}$$

This implies that there exists $T_4 > 0$ depending on $\mathcal{P}_0, \mathcal{P}_1$ so that for $t \in [0, \min(T, T_n)]$ with $T = \min(T_1, T_2, T_3, T_4)$,

$$\mathcal{E}^{n+1}(t) \leq \mathcal{P}(\mathcal{E}_0) + \mathcal{P}(E_1, E_2, E_4) + \mathcal{P}_1 + \frac{1}{4} \mathcal{P}_0.$$

Since E_1, E_2, E_3 are independent of \mathcal{P}_0 , we can take $\mathcal{P}_0 = 4(\mathcal{P}(\mathcal{E}_0) + \mathcal{P}(E_1, E_2, E_4) + \mathcal{P}_1)$. Thus, we deduce that for $t \in [0, \min(T, T_n)]$,

$$\mathcal{E}^{n+1}(t) \leq \frac{\mathcal{P}_0}{2}.$$

This completes the proof of uniform estimates. \square

12. CAUCHY SEQUENCE AND THE LIMIT SYSTEM

This section is devoted to showing that the approximate sequence constructed in last section is a Cauchy sequence.

12.1. Set-up. According to the uniform estimates of the approximate sequence, we may assume that

$$\sum_{k=0}^2 \left(\|\partial_t^k(V^n, B^n, V_b^n, V_1^n, B_1^n, V_{b,1}^n)\|_{H^{s-\frac{1}{2}-k}} + \|\partial_t^k \eta^n\|_{H^{s+\frac{1}{2}-k}} + \|\partial_t^k \eta_1^n\|_{H^{s-\frac{1}{2}-k}} \right. \\ \left. + \|\partial_t^k \widetilde{\omega}^n\|_{H^{s-1-k}(\overline{\mathcal{S}})} \right) \leq \mathcal{P}_0.$$

Here and in what follows we denote by \mathcal{P}_0 a constant depending only on $\|v_0\|_{H^s(\Omega_0)}$, $\|\eta_0\|_{H^{s+\frac{1}{2}}}$ and h_0, c_0 , which may change from line to line. For a function $f(x, y)$ defined on $\{(x, y) : -1 < y < \eta(x)\}$, we denote $\widetilde{f}(x, z) \triangleq f(x, \rho_\delta(x, z))$, where $\rho_\delta(x, z) = z + (1+z)e^{\delta z|D|\eta}$.

We introduce

$$\delta_{\mathcal{E}^n}(t) \stackrel{\text{def}}{=} \delta_{\mathcal{E}_1^n}(t) + \delta_{\mathcal{E}_2^n}(t),$$

where $\delta_{\mathcal{E}_1^n}(t)$ and $\delta_{\mathcal{E}_2^n}(t)$ are given by

$$\delta_{\mathcal{E}_1^n}(t) = \|D_t \delta_{\zeta^n}(t)\|_{H^{s-2}} + \|T_{\sqrt{a^n \lambda^n}} \delta_{\zeta^n}(t)\|_{H^{s-2}}, \\ \delta_{\mathcal{E}_2^n}(t) = \|(\delta_{V^n}, \delta_{B^n}, \delta_{V_b^n}, \delta_{V_1^n}, \delta_{B_1^n}, \delta_{V_{b,1}^n}, \delta_{\eta_1^n}, \delta_{\eta^n})(t)\|_{H^{s-\frac{3}{2}}} + \|\delta_{\widetilde{\omega}^n}(t)\|_{H^{s-2}(\overline{\mathcal{S}})}.$$

Here $D_t = \partial_t + T_{V^{n+1}} \cdot \nabla$ and we denote $\delta_{f^n} \triangleq f^{n+1} - f^n$.

Throughout this section, we denote by $L_i^n (i = 1, 2)$ some nonlinear terms, which satisfy

$$\|L_i^n(t)\|_{H^{s-\frac{1}{2}-i}} \leq \mathcal{P}_0(\delta_{\mathcal{E}^n}(t) + \delta_{\mathcal{E}^{n-1}}(t)).$$

Using Lemma 2.10 and Proposition 2.6, it is easy to find that

$$\begin{cases} D_t(\delta_{V^n}, \delta_{B^n}) = L_1^n + L(\delta_{a^{n-1}}), \\ (\partial_t + V_b^n \cdot \nabla) \delta_{V_b^n} = -\delta_{\nabla P^{n-1}|_{y=-1}}, \\ D_t \delta_{\zeta^n} = L_2^n + L_3^n + \delta_{F_2}, \\ (\partial_t + T_{V^{n+1}} \cdot \nabla)(\delta_{V_1^n}, \delta_{B_1^n}) = L_1^n + L(\delta_{a^{n-1}}), \\ (\partial_t + V_b^{n+1} \cdot \nabla) \delta_{V_{b,1}^n} = -\delta_{\nabla P^{n-1}|_{y=-1}}, \\ (\partial_t + V^{n+1} \cdot \nabla) \delta_{\eta_1^n} = L_1^n, \end{cases} \quad (12.1)$$

where $L(\delta_{a^{n-1}})$ is a nonlinear term satisfying

$$\|L(\delta_{a^{n-1}})\|_{H^{s-\frac{3}{2}}} \leq C \|\delta_{a^{n-1}}\|_{H^{s-\frac{3}{2}}},$$

and L_3^n is given by

$$L_3^n = (R(\eta_1^n) - R(\eta_1^{n-1}))V_1^n + \zeta^n(R(\eta_1^n) - R(\eta_1^{n-1}))B_1^n + (R_\omega^n - R_\omega^{n-1}).$$

12.2. Elliptic estimates with a parameter. In order to compare the solution of the elliptic equations in two different domains and with different boundary values, we consider the following elliptic equation in a domain $\Omega_\tau(t) = \{(x, y) : -1 < y < \eta(\tau, t, x), x \in \mathbf{R}^d\}$ with a parameter $\tau \in [0, 1]$:

$$\begin{cases} \Delta_{x,y} v(\tau, t, x, y) = F(\tau, t, x) & \text{in } \Omega_\tau, \\ v|_{y=\eta(\tau, t, x)} = V(\tau, t, x), \quad v|_{y=-1} = V^b(\tau, t, x). \end{cases} \quad (12.2)$$

In the sequel, we denote $f_\tau \triangleq \partial_\tau f(\tau, t, x, y)$. Then v_τ satisfies

$$\begin{cases} \Delta_{x,y} v_\tau = F_\tau & \text{in } \Omega_\tau, \\ v_\tau|_{y=\eta(\tau,t,x)} = V_\tau - \partial_y v \eta_\tau, & v_\tau|_{y=-1} = V_\tau^b. \end{cases}$$

If (η, V, V^b) and F satisfy

$$\begin{aligned} \|\eta\|_{H^{s-\frac{1}{2}}} + \|V\|_{H^{s-\frac{1}{2}}} + \|V_\tau^b\|_{H^{s-\frac{1}{2}}} + \|\widetilde{F}_\tau\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} &\leq C, \\ \|\eta_\tau\|_{H^{s-\frac{3}{2}}} + \|V_\tau\|_{H^{s-\frac{3}{2}}} + \|V_\tau^b\|_{H^{s-\frac{3}{2}}} + \|\widetilde{F}_\tau\|_{H^{s-\frac{3}{2}}(\overline{\mathcal{S}})} &\leq C, \\ \eta(\tau, t, x) + 1 &\geq \frac{h_0}{2} \quad \text{for } (\tau, t, x) \in [0, 1] \times [0, T] \times \mathbf{R}^d, \end{aligned}$$

then we infer from Proposition 4.9 and Proposition 4.10 that for any $\sigma \in [-\frac{1}{2}, s - \frac{5}{2}]$,

$$\|\nabla_{x,z} \widetilde{v}_\tau\|_{X^\sigma([-\frac{1}{2}, 0])} \leq C(\|\widetilde{F}_\tau\|_{L^2(H^{\sigma-\frac{1}{2}}(\overline{\mathcal{S}}))} + \|(V_\tau, \eta_\tau)\|_{H^{\sigma+1}} + \|V_\tau^b\|_{H^{\frac{1}{2}}}), \quad (12.3)$$

$$\|\nabla_{x,z} \widetilde{v}_\tau\|_{H^{\sigma+\frac{1}{2}}(\overline{\mathcal{S}})} \leq C(\|\widetilde{F}_\tau\|_{H^{\sigma-\frac{1}{2}}(\overline{\mathcal{S}})} + \|(V_\tau, V_\tau^b, \eta_\tau)\|_{H^{\sigma+1}}). \quad (12.4)$$

Now we take in the elliptic equation (12.2):

$$\begin{aligned} \eta &= \tau \eta_1^{n+1} + (1-\tau) \eta_1^n, \\ V &= \tau(V^{n+1}, B^{n+1}) + (1-\tau)(V^n, B^n), \\ V^b &= \tau(V_b^{n+1}, 0) + (1-\tau)(V_b^n, 0), \\ F &= \nabla_{x,y} \cdot (\tau \widetilde{\omega}^{n+1} \circ \Phi^{-1} + (1-\tau) \widetilde{\omega}^n \circ \Phi^{-1}), \end{aligned}$$

where $\Phi(\tau, x, z) = (x, \rho_\delta(\tau, x, z))$ with $\rho_\delta(\tau, x, z) = z + (1+z)e^{\delta z|D|}\eta(\tau, t, \cdot)$. Notice that

$$\begin{aligned} \delta_{\widetilde{v}^n} &= \widetilde{v}^{n+1} - \widetilde{v}^n = \int_0^1 \widetilde{v}_\tau d\tau, \quad \eta_\tau = \delta_{\eta^n}, \quad V_\tau = \delta_{(V^n, B^n)}, \quad V_\tau^b = \delta_{(V_b^n, 0)}, \\ \|\widetilde{F}_\tau\|_{H^{s-3}(\overline{\mathcal{S}})} &\leq \mathcal{P}_0 \delta_{\mathcal{E}_2^n}(t). \end{aligned}$$

Then we can deduce from (12.4) that

$$\|\nabla_{x,z} \delta_{\widetilde{v}^n}\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^n}(t). \quad (12.5)$$

In a similar way, we can deduce that

$$\|\nabla_{x,z} \delta_{\widetilde{v}_1^n}\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^n}(t). \quad (12.6)$$

Let $\omega(\tau, t, x) = \tau \widetilde{\omega}^{n+1} \circ \Phi^{-1} + (1-\tau) \widetilde{\omega}^n \circ \Phi^{-1}$ and

$$\widetilde{\mathcal{D}}_t = \partial_t + (\widetilde{v}_2^{n+1} \circ \Phi^{-1} - \partial_t \Phi^{-1}) \cdot \mathcal{A} \nabla_{x,y}$$

with $\mathcal{A} = (\partial_i \Phi_j^{-1})^{-1}$. Then ω_τ satisfies

$$\|(\widetilde{\mathcal{D}}_t \omega_\tau) \circ \Phi\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^n}(t). \quad (12.7)$$

This in turn implies that

$$\|\delta_{\widetilde{\mathcal{D}}_t v_1^n}\|_{H^{s-1}(\overline{\mathcal{S}})} + \|(\widetilde{\mathcal{D}}_t^2 \omega_\tau) \circ \Phi\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^n}(t). \quad (12.8)$$

Similarly, there holds for $\delta_{v_\omega^{n-1}}$,

$$\|\delta_{v_\omega^{n-1}}\|_{H^{s-1}(\overline{\mathcal{S}})} + \|\delta_{\widetilde{\mathcal{D}}_t v_\omega^{n-1}}\|_{X^{s-\frac{3}{2}}([-\frac{1}{2}, 0])} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^{n-1}}(t), \quad (12.9)$$

$$\|\delta_{\widetilde{\mathcal{D}_t^2 v_\omega^{n-1}}}\|_{X^{s-\frac{3}{2}}([-\frac{1}{2}, 0])} \leq \mathcal{P}_0 \delta_{\mathcal{E}^{n-1}}(t). \quad (12.10)$$

To compare the pressure, we consider

$$\begin{cases} -\Delta_{x,y} P(\tau, t, x, y) = F(\tau, t, x) & \text{in } \Omega_\tau, \\ P|_{y=\eta(\tau, t, x)} = 0, \quad \partial_y P|_{y=-1} = -1. \end{cases}$$

where $\eta(\tau, t, x) = \tau\eta^n + (1-\tau)\eta^{n-1}$ and

$$F = ((\partial_i(v_1^j)^n \partial_j(v_1^i)^n) \circ \Phi_1^n - (\partial_i(v_1^j)^{n-1} \partial_j(v_1^i)^{n-1}) \circ \Phi_1^{n-1}) \circ \Phi^{-1}.$$

Then P_τ satisfies

$$\begin{cases} -\Delta_{x,y} P_\tau = F_\tau & \text{in } \Omega_\tau, \\ P_\tau|_{y=\eta(\tau, t, x)} = -\partial_y P_\tau, \quad \partial_y P_\tau|_{y=-1} = 0. \end{cases}$$

Thus, we can deduce by a similar proof of Lemma 11.6 that

$$\|\delta_{a^{n-1}}\|_{H^{s-\frac{5}{2}}} + \|\delta_{\nabla P^{n-1}|_{y=-1}}\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^{n-1}}(t), \quad (12.11)$$

$$\|\delta_{a^{n-1}}\|_{H^{s-\frac{3}{2}}} + \|\delta_{\nabla P^{n-1}|_{y=-1}}\|_{H^{s-\frac{3}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}^{n-1}}(t), \quad (12.12)$$

$$\|\delta_{\nabla \partial_t P^{n-1}|_{y=-1}}\|_{H^{s-\frac{5}{2}}} + \|\delta_{\partial_t a^{n-1}}\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}^{n-1}}(t). \quad (12.13)$$

To compare the remainder of (DN) operators, we take in (12.2):

$$\eta = \tau\eta_1^n + (1-\tau)\eta_1^{n-1}, \quad V = (V_1^n, B_1^n), \quad V^b = 0, \quad F = 0.$$

Then v_τ satisfies

$$\begin{cases} \Delta_{x,y} v_\tau = 0 & \text{in } \Omega_\tau, \\ v_\tau|_{y=\eta(\tau, t, x)} = -\partial_y v_\tau, \quad v_\tau|_{y=-1} = 0. \end{cases}$$

Notice that

$$\|D_t^u \eta_\tau\|_{H^{s-\frac{3}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^{n-1}}(t), \quad \|(D_t^u)^2 \eta_\tau\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}^{n-1}}(t).$$

Here $D_t^u = \partial_t + V^n \cdot \nabla$. Thus, we can deduce from a similar proof of Proposition 11.7 that

$$\|(R(\eta_1^n) - R(\eta_1^{n-1}))(V_1^n, B_1^n)\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^{n-1}}(t), \quad (12.14)$$

$$\|D_t^u(R(\eta_1^n) - R(\eta_1^{n-1}))(V_1^n, B_1^n)\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^{n-1}}(t), \quad (12.15)$$

$$\|(D_t^u)^2(R(\eta_1^n) - R(\eta_1^{n-1}))(V_1^n, B_1^n)\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}^{n-1}}(t). \quad (12.16)$$

12.3. Energy estimates. Following the proofs in section 11.7 and using the estimates in subsection 12.4, we can deduce that

$$\|\delta_{f_1^{n-1}}\|_{H^{s-2}} + \|\delta_{f_3^{n-1}}\|_{H^{s-\frac{3}{2}}} \leq \mathcal{P}_0 (\delta_{\mathcal{E}^n}(t) + \delta_{\mathcal{E}^{n-1}}(t)),$$

$$\|\delta_{f_3^{n-1}}\|_{H^{s-2}} \leq \mathcal{P}_0 (\|\delta_{\eta^{n-1}}\|_{H^{s-1}} + \delta_{\mathcal{E}_2^{n-1}}(t)),$$

$$\|\delta_{f_2^{n-1}}\|_{H^{s-\frac{5}{2}}} + \|D_t^u \delta_{f_4^{n-1}}\|_{H^{s-\frac{5}{2}}} + \|D_t^u \delta_{f_\omega^{n-1}}\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 (\delta_{\mathcal{E}_2^{n-1}}(t) + \delta_{\mathcal{E}_2^n}(t)),$$

$$\|D_t \delta_{f_2^{n-1}}\|_{H^{s-\frac{5}{2}}} + \|(D_t^u)^2 \delta_{f_4^{n-1}}\|_{H^{s-\frac{5}{2}}} + \|(D_t^u)^2 \delta_{f_\omega^{n-1}}\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 (\delta_{\mathcal{E}^n}(t) + \delta_{\mathcal{E}^{n-1}}(t)).$$

By (12.14) and (12.9), we have

$$\|L_3^n\|_{H^{s-\frac{5}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^{n-1}}(t).$$

We also have

$$D_t^2 \delta_{\zeta^n} + T_{a^n \lambda^n} \delta_{\zeta^n} = L_2^n + L_4^n, \quad (12.17)$$

where L_4^n is given by

$$L_4^n = (f_2^n + D_t f_3^n + D_t f_4^n + D_t f_\omega^n) - (f_2^{n-1} + D_t f_3^{n-1} + D_t f_4^{n-1} + D_t f_\omega^{n-1}).$$

For $\delta_{\widetilde{\omega}^n}$, we have

$$\|(\partial_t + \overline{v}^{n+1} \cdot \nabla_{x,z}) \delta_{\widetilde{\omega}^n}\|_{H^{s-2}(\overline{\mathcal{S}})} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2^{n-1}}(t).$$

Making the energy estimates for the system (12.1), we obtain

$$\delta_{\mathcal{E}_2^n}(t)^2 \leq \mathcal{P}_0 \int_0^t (\delta_{\mathcal{E}^n}(t') + \delta_{\mathcal{E}^{n-1}}(t'))^2 dt'.$$

While, making the energy estimates for the system (12.17), we obtain

$$\begin{aligned} \delta_{\mathcal{E}_1^n}(t)^2 &\leq \mathcal{P}_0 \int_0^t (\delta_{\mathcal{E}^n}(t') + \delta_{\mathcal{E}^{n-1}}(t'))^2 dt' + \frac{1}{2} \delta_{\mathcal{E}^{n-1}}(t)^2 \\ &\quad + \mathcal{P}_0 (\delta_{\mathcal{E}_2^n}(t) + \delta_{\mathcal{E}_2^{n-1}}(t))^2. \end{aligned}$$

Then an induction argument ensures that there exists $T > 0$ depending only on \mathcal{P}_0 so that for $t \in [0, T]$,

$$\delta_{\mathcal{E}^n}(t) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

This shows that the approximate sequence is a Cauchy sequence.

12.4. The limit system. Let $(V, B, V_b, V_1, B_1, V_{b,1}, \zeta, \eta_1, \eta, \omega, v, v_1, P)$ be the limit of the Cauchy sequence

$$(V^n, B^n, V_b^n, V_1^n, B_1^n, V_{b,1}^n, \zeta^n, \eta^n, \eta_1^n, \omega^n, v^n, v_1^n, P^n).$$

Taking the limit for the approximate system (10.1)–(10.8), we obtain the following limit system: The boundary velocity (V, B, V_b) and ζ satisfy

$$\begin{cases} D_t V = T_\zeta a + T_a \zeta + R(\zeta, a) + (T_{V_1} - V_1) \cdot \nabla V, \\ D_t B = a - 1 + (T_{V_1} - V_1) \cdot \nabla B, \\ (\partial_t + V_b \cdot \nabla) V_b = -\nabla P|_{y=-1}, \\ D_t \zeta = T_\lambda (V + T_\zeta B) + (T_{V_1} - V_1) \cdot \zeta + [T_\zeta, T_\lambda] B_1 \\ \quad + (\zeta - T_\zeta) T_\lambda B + R(\eta_1) V_1 + \zeta R(\eta_1) B_1 + R_\omega, \end{cases} \quad (12.18)$$

where $D_t = \partial_t + T_V \cdot \nabla$, $a = -\partial_y P|_{y=\eta}$, $\lambda = \lambda(\eta_1)$, and

$$\begin{aligned} (R_\omega)^i &= (\partial_y (v_\omega)^i - \partial_{x_j} (v_\omega)^i \cdot \partial_{x_j} \eta_1) + \partial_{x_i} \eta_1 (\partial_y (v_\omega)^{d+1} - \partial_{x_j} \eta_1 \partial_{x_j} (v_\omega)^{d+1}) \\ &\quad + (\omega_{i,d+1} - \partial_{x_j} \eta_1 \omega_{ij} + \partial_{x_i} \eta_1 \partial_{x_j} \eta_1 \omega_{j,d+1})|_{y=\eta_1} \end{aligned}$$

with v_ω given by

$$\begin{cases} -\Delta_{x,y} v_\omega = \nabla_{x,y} \times \omega & \text{in } \Omega_t = \{(x, y) : -1 < y < \eta(t, x)\}, \\ v_\omega|_{y=\eta_1} = 0, \quad v_\omega|_{y=-1} = (V_b, 0), \end{cases}$$

and $(V_1, B_1, V_{b,1})$ satisfies

$$\begin{cases} D_t (V_1, B_1) = D_t (V, B), \\ (\partial_t + V_b \cdot \nabla) V_{b,1} = -\nabla P|_{y=-1}. \end{cases} \quad (12.19)$$

The free surface (η, η_1) satisfies

$$-\Delta \eta + \eta = -\operatorname{div} \zeta + \eta_1, \quad (12.20)$$

$$(\partial_t + V \cdot \nabla) \eta_1 = B_1. \quad (12.21)$$

The vorticity ω satisfies

$$\partial_t \omega + (v^h \cdot \nabla + v_1^{d+1} \partial_y) \omega = \omega \cdot \nabla_{x,y} v_1, \quad (12.22)$$

where the velocity (v, v_1) is given by

$$\begin{cases} -\Delta_{x,y} v = \nabla_{x,y} \times \omega & \text{in } \Omega_t, \\ v|_{y=\eta_1} = (V, B), \quad v|_{y=-1} = (V_b, 0), \end{cases} \quad (12.23)$$

and

$$\begin{cases} -\Delta_{x,y} v_1 = \nabla_{x,y} \times \omega & \text{in } \Omega_t, \\ v_1|_{y=\eta_1} = (V_1, B_1), \quad v_1|_{\Gamma_b} = (V_{b,1}, 0). \end{cases} \quad (12.24)$$

Let $\tilde{\Omega}_t = \{(x, y) : y = \eta(t, x)\}$. The pressure P satisfies

$$\begin{cases} -\Delta_{x,y} P = (\partial_i(v_1^j) \partial_j(v_1^i)) \circ \Phi_1 \circ (\Phi)^{-1} & \text{in } \tilde{\Omega}_t, \\ P|_{y=\eta} = 0, \quad \partial_y P|_{y=-1} = 0, \end{cases} \quad (12.25)$$

where $\Phi(x, z) = (x, \rho_{\delta, \eta}(x, z))$ and $\Phi_1(x, z) = (x, \rho_{\delta, \eta_1}(x, z))$ with $\rho_{\delta, \eta}(x, z) = z + (1 + z)e^{\delta z|D|\eta}$.

13. FROM THE LIMIT SYSTEM TO THE EULER EQUATIONS

The goal of this section is to show that the limit system (12.18)–(12.25) is equivalent to the Euler equations (1.1)–(1.5).

First of all, it follows from (12.19) and the third equation of (10.1) that

$$(V, B, V_b) = (V_1, B_1, V_{b,1}).$$

Hence, $v = v_1$ since they satisfy the same elliptic equation with the same boundary conditions. Thus, we deduce from (12.18) that

$$\begin{cases} (\partial_t + V \cdot \nabla) V = a \zeta, \quad a = \nabla P|_{y=\eta}, \\ (\partial_t + V \cdot \nabla) B = a - 1, \\ (\partial_t + V_b \cdot \nabla) V_b = -\nabla P|_{y=-1}, \\ (\partial_t + V \cdot \nabla) \zeta = G(\eta_1) V + \zeta G(\eta_1) B + R_\omega. \end{cases} \quad (13.1)$$

For the free surface, we have

$$(\partial_t + V \cdot \nabla) \eta_1 = B. \quad (13.2)$$

The vorticity ω satisfies

$$\begin{cases} \partial_t \omega + v \cdot \nabla_{x,y} \omega = \omega \cdot \nabla_{x,y} v & \text{in } \Omega_t, \\ -\Delta_{x,y} v = \nabla_{x,y} \times \omega & \text{in } \Omega_t, \\ v|_{y=\eta_1} = (V, B), \quad v|_{y=-1} = (V_b, 0). \end{cases} \quad (13.3)$$

It remains to show that

$$\eta = \eta_1, \quad \omega = \nabla_{x,y} \times v, \quad \operatorname{div} v = 0. \quad (13.4)$$

For this end, we introduce

$$\begin{aligned} G &= v_t + v \cdot \nabla_{x,y} v + (\nabla_{x,y}(P + gy)) \circ \overline{\Phi}, \\ \delta_\omega &= \omega - \overline{\omega}, \quad \delta_d = \operatorname{div} v, \quad \delta_\zeta = \zeta - \nabla \eta_1. \end{aligned}$$

where $\overline{\omega} = \nabla_{x,y} \times v$ and $\overline{\Phi} = \Phi \circ (\Phi_1)^{-1}$.

In what follows, we denote by $L(\cdot)$ a linear function, which may be different from line to line.

Lemma 13.1. *It holds that*

$$\bar{\Phi}(x, y) = (x, h(x, y)),$$

where $h(x, y)$ satisfies

$$\|\nabla_{x,y}(h - y)\|_{H^1(\Omega_t)} \leq C\|\eta - \eta_1\|_{H^{\frac{3}{2}}}.$$

Proof. Let $\Phi_1^{-1}(x, y) = (x, z(x, y))$, i.e.,

$$y = z(x, y) + (1 + z(x, y))e^{\delta z(x, y)|D|}\eta_1.$$

Hence, we have

$$\bar{\Phi}(x, y) = (x, z(x, y) + (1 + z(x, y))e^{\delta z(x, y)|D|}\eta) \triangleq (x, h(x, y)).$$

Using the fact that

$$\begin{aligned} \nabla z(x, y)(1 + e^{\delta z(x, y)|D|}\eta_1) + (1 + z(x, y))(\nabla z(x, y)e^{\delta z(x, y)|D|}|D|\eta_1 + e^{\delta z(x, y)|D|}\nabla\eta_1) &= 0, \\ \partial_y z(x, y)(1 + e^{\delta z(x, y)|D|}\eta_1) + (1 + z(x, y))\partial_y z(x, y)e^{\delta z(x, y)|D|}|D|\eta_1 &= 1, \end{aligned}$$

we deduce that

$$\nabla_{x,y}(h(x, y) - y) = L(e^{\delta z(x, y)|D|}(\eta - \eta_1), e^{\delta z(x, y)|D|}\nabla(\eta - \eta_1)),$$

which implies that

$$\begin{aligned} \|\nabla_{x,y}(h - y)\|_{H^1(\Omega_t)} &\leq C\|e^{\delta z(x, y)|D|}(\eta - \eta_1)\|_{H^1(\Omega_t)} + C\|e^{\delta z(x, y)|D|}\nabla(\eta - \eta_1)\|_{H^1(\Omega_t)} \\ &\leq C\|e^{\delta z|D|}(\eta - \eta_1)\|_{L^2_z(-1, 0; H^2)} \leq C\|\eta - \eta_1\|_{H^{\frac{3}{2}}}. \end{aligned}$$

The proof is finished. \square

Let us first derive the equation of G .

Lemma 13.2. *It holds that*

$$\begin{cases} \Delta_{x,y}G = L(\delta_d, \nabla_{x,y}\delta_d, \nabla_{x,y}^2\delta_d, \delta_\omega, \nabla_{x,y}\delta_\omega, \nabla_{x,y}^2\delta_\omega, \nabla_{x,y}(h - y), \nabla_{x,y}^2h), \\ G|_{y=\eta_1} = 0, \quad G|_{y=-1} = 0. \end{cases}$$

In particular, we have

$$\|G\|_{H^2(\Omega_t)} \leq C(\|(\delta_d, \delta_\omega)\|_{L^2(\Omega_t)} + \|\nabla_{x,y}(h - y)\|_{H^1(\Omega_t)}).$$

Proof. Thanks to (13.3), we have

$$\Delta_{x,y}v_t = -\nabla_{x,y} \times \omega_t = \nabla_{x,y} \times (v \cdot \nabla_{x,y}\omega - \omega \cdot \nabla_{x,y}v).$$

A direct calculation gives

$$\begin{aligned} \Delta_{x,y}(v \cdot \nabla_{x,y}v) &= -\nabla_{x,y} \times (\nabla_{x,y} \times (v \cdot \nabla v)) + \nabla_{x,y}\operatorname{div}(v \cdot \nabla_{x,y}v) \\ &= -\nabla_{x,y} \times (v \cdot \nabla_{x,y}\bar{\omega} - \bar{\omega} \cdot \nabla_{x,y}v + \bar{\omega}\delta_d) \\ &\quad + \nabla_{x,y}(\partial_j v^i \partial_i v^j) + \nabla_{x,y}(v \cdot \nabla_{x,y}\delta_d), \end{aligned}$$

Using (12.25) and Lemma 13.1, we obtain

$$\Delta_{x,y}(\nabla_{x,y}P \circ \bar{\Phi}) = -\nabla_{x,y}(\partial_j v^i \partial_i v^j) + L(\nabla_{x,y}(h - y), \nabla_{x,y}^2h).$$

The first equation of the lemma follows by summing up the above equations. The boundary condition of G follows from (13.1). \square

Now we have

$$v_t + v \cdot \nabla_{x,y} v + (\nabla_{x,y}(P + gy)) \circ \bar{\Phi} = G. \quad (13.5)$$

Take the divergence to get

$$\partial_t \delta_d + v \cdot \nabla_{x,y} \delta_d = \operatorname{div}_{x,y} G + L(\nabla_{x,y}(h - y)),$$

which along with Lemma 13.1 and Lemma 13.2 implies that

$$\begin{aligned} \frac{d}{dt} \|\delta_d\|_{H^1(\Omega_t)}^2 &\leq C(\|\delta_d\|_{H^1(\Omega_t)}^2 + \|G\|_{H^2(\Omega_t)} + \|\nabla_{x,y}(h - y)\|_{H^1(\Omega_t)}) \\ &\leq C(\|(\delta_d, \delta_\omega)\|_{H^1(\Omega_t)}^2 + \|\eta - \eta_1\|_{H^{\frac{3}{2}}}). \end{aligned} \quad (13.6)$$

Taking the curl to (13.5), we obtain

$$\partial_t \bar{\omega} + v \cdot \nabla_{x,y} \bar{\omega} = \bar{\omega} \cdot \nabla_{x,y} v + \nabla_{x,y} \times G + L(\nabla_{x,y}(h - y)),$$

from which and (13.3), we infer

$$\partial_t \delta_\omega + v \cdot \nabla_{x,y} \delta_\omega = \delta_\omega \cdot \nabla_{x,y} v + \nabla_{x,y} \times G + L(\nabla_{x,y}(h - y)).$$

Then similar to (13.6), we have

$$\frac{d}{dt} \|\delta_\omega\|_{H^1(\Omega_t)}^2 \leq C(\|(\delta_d, \delta_\omega)\|_{H^1(\Omega_t)}^2 + \|\eta - \eta_1\|_{H^{\frac{3}{2}}}). \quad (13.7)$$

Let $\zeta_1 = \nabla \eta_1$. By a similar derivation of (6.10), we get

$$\partial_t \zeta_1 + V \cdot \nabla \zeta_1 = G(\eta_1)V + \zeta_1 G(\eta_1)B + R_{\bar{\omega}},$$

where $R_{\bar{\omega}}$ is given by

$$\begin{aligned} (R_{\bar{\omega}})^i &= (\partial_y(v_\omega))^i - \partial_{x_j}(v_\omega)^i \cdot \partial_{x_j} \eta_1 + \partial_{x_i} \eta_1 (\partial_y(v_\omega))^{d+1} - \partial_{x_j} \eta_1 \partial_{x_j}(v_\omega)^{d+1} \\ &\quad + (\bar{\omega}_{i,d+1} - \partial_{x_j} \eta_1 \bar{\omega}_{ij} + \partial_{x_i} \eta_1 \partial_{x_j} \eta_1 \bar{\omega}_{j,d+1}) \Big|_{y=\eta_1}. \end{aligned}$$

Hence, δ_ζ satisfies

$$\partial_t \delta_\zeta + V \cdot \nabla \delta_\zeta = \delta_\zeta G(\eta_1)B + R_\omega - R_{\bar{\omega}}.$$

This ensures that

$$\frac{d}{dt} \|\delta_\zeta\|_{H^{\frac{1}{2}}}^2 \leq C(\|\delta_\zeta\|_{H^{\frac{1}{2}}}^2 + \|\delta_\omega\|_{H^1(\Omega_t)}^2). \quad (13.8)$$

On the other hand, we know

$$-\Delta(\eta - \eta_1) + (\eta - \eta_1) = \operatorname{div}(\zeta - \zeta_1) = \operatorname{div} \delta_\zeta,$$

which implies

$$\|\eta - \eta_1\|_{H^{\frac{3}{2}}} \leq C\|\delta_\zeta\|_{H^{\frac{1}{2}}}.$$

Then we deduce from (13.6)–(13.8) that

$$\frac{d}{dt} (\|\delta_d\|_{H^1(\Omega_t)}^2 + \|\delta_\omega\|_{H^1(\Omega_t)}^2 + \|\delta_\zeta\|_{H^{\frac{1}{2}}}^2) \leq C(\|\delta_d\|_{H^1(\Omega_t)}^2 + \|\delta_\omega\|_{H^1(\Omega_t)}^2 + \|\delta_\zeta\|_{H^{\frac{1}{2}}}^2)$$

together with the initial condition

$$\delta_d|_{t=0} = 0, \quad \delta_\omega|_{t=0} = 0, \quad \delta_\zeta|_{t=0} = 0.$$

Gronwall's inequality implies that $\delta_d = 0, \delta_\omega = 0, \delta_\zeta = 0$. This proves (13.4). \square

14. PROOF OF THEOREM 1.1

This section is devoted to proving the local well-posedness of the system (1.1)–(1.5) for the low regularity initial data.

14.1. Construction of approximate smooth solution. First of all, we can construct a sequence of smooth functions η_0^n so that

$$\begin{aligned} \|\eta_0^n - \eta_0\|_{H^{s+\frac{1}{2}}} &\longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \\ \eta_0^n(x) + 1 &\geq \frac{3}{4}h_0 \quad \text{for } x \in \mathbf{R}^d. \end{aligned}$$

Let $\tilde{v}_0(x, z) = v_0 \circ \Phi(x, z)$, where $\Phi(x, z) = (x, \rho_\delta(x, z))$ with $\rho_\delta(x, z) = z + (1+z)e^{\delta z|D|}\eta_0$. Then we take a sequence of smooth functions \tilde{v}_0^n so that

$$\|\tilde{v}_0^n - \tilde{v}_0\|_{H^{s+\frac{1}{2}}(\overline{\mathcal{S}})} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

Let $v_0^n(x, y) = \tilde{v}_0^n \circ (\Phi^n)^{-1}(x, y)$, where $\Phi^n(x, z) = (x, \rho_\delta^n(x, z))$ with $\rho_\delta^n(x, z) = z + (1+z)e^{\delta z|D|}\eta_0^n$.

The pressure P_0^n associated with the initial data (v_0^n, η_0^n) is defined by

$$\begin{cases} -\Delta_{x,y} P_0^n = \partial_i (v_0^n)^j \partial_j (v_0^n)^i & \text{in } \Omega_0^n, \\ P_0^n|_{y=\eta_0^n(x)} = 0, \quad \partial_y P_0^n|_{y=-1} = -1, \end{cases}$$

where $\Omega_0^n = \{(x, y) : y = \eta_0^n(x), x \in \mathbf{R}^d\}$. Then we can show that

$$\|\nabla_{x,z}(P_0^n \circ \Phi^n - P_0 \circ \Phi)\|_{X^{s-\frac{1}{2}}([-\frac{1}{2}, 0])} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

This ensures that for n big enough, there holds

$$-\partial_y P_0^n|_{y=\eta_0^n} \geq \frac{3}{4}c_0.$$

Thus, we apply Theorem 10.1 to obtain a sequence of smooth solutions (v^n, η^n, P^n) associated with the initial data (v_0^n, η_0^n) on a maximal existence time interval $[0, T_n)$.

14.2. Uniform estimates and existence. We denote

$$(V^n, B^n) = v^n|_{y=\eta^n(t,x)}, \quad V_b^n = (v^n)^h|_{y=-1}.$$

We define

$$\begin{aligned} E_s^n(t) &\stackrel{\text{def}}{=} \|(V^n, B^n, V_b^n)(t)\|_{H^s} + \|\eta^n(t)\|_{H^{s+\frac{1}{2}}} + \|\tilde{\omega}^n(t)\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}, \\ E_{s,l}^n(t) &\stackrel{\text{def}}{=} \|(V^n, B^n)(t)\|_{H^{s-\frac{1}{2}}}^2 + \|V_b^n(t)\|_{H^s}^2 + \|\eta^n(t)\|_{H^s}^2 + \|\tilde{\omega}^n(t)\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}. \end{aligned}$$

The goal of this subsection is to show that there exists $T > 0$ depending only on $E_s(0)$ and c_0, h_0, s such that for any $t \in [0, \min(T, T_n))$, there holds

$$E_s^n(t) \leq \mathcal{P}(E_s(0)). \quad (14.1)$$

Here and in what follows we denote by \mathcal{P} an increasing function depending only on c_0, h_0, s , which may change from line to line.

First of all, it follows from Proposition 9.3 that

$$\frac{d}{dt}(\|(V^n, B^n)\|_{H^{s-\frac{1}{2}}}^2 + \|V_b^n\|_{H^s}^2 + \|\eta^n\|_{H^s}^2) \leq \mathcal{P}(E_s^n(t)). \quad (14.2)$$

Next, we estimate the vorticity ω^n , which satisfies

$$\partial_t \widetilde{\omega}^n + \overline{v}^n \cdot \nabla_{x,z} \widetilde{\omega}^n = \widetilde{\omega}^{n,h} \cdot (\nabla \widetilde{v}^n - \frac{\nabla \rho_\delta^n}{\partial_z \rho_\delta^n} \partial_z \widetilde{v}^n) + \widetilde{\omega}^{n,d+1} \frac{\partial_z \widetilde{v}^n}{\partial_z \rho_\delta^n} \triangleq F^n,$$

where $\overline{v}^n = (\widetilde{v}^n, \frac{1}{\partial_z \rho_\delta^n} (\widetilde{v}^n_{d+1} - \partial_t \rho_\delta^n - \widetilde{v}^{n,h} \cdot \nabla \rho_\delta^n))$. By Lemma 9.6 and Lemma 2.13, we have

$$\|\overline{v}^n\|_{H^{s+\frac{1}{2}}(\overline{\mathcal{S}})} + \|F^n\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} \leq \mathcal{P}(E_s^n(t)).$$

Let \overline{v}_e^n and F_e^n be the extension of \overline{v}^n and F^n to \mathbf{R}^{d+1} so that

$$\begin{aligned} \|\overline{v}_e^n\|_{H^{s+\frac{1}{2}}(\mathbf{R}^{d+1})} &\leq C \|\overline{v}^n\|_{H^{s+\frac{1}{2}}(\overline{\mathcal{S}})}, \\ \|F_e^n\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{d+1})} &\leq C \|F^n\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})}. \end{aligned}$$

We define $\widetilde{\omega}_e^n$ to be a solution of the following transport equation in \mathbf{R}^{d+1}

$$\partial_t \widetilde{\omega}_e^n + \overline{v}_e^n \cdot \nabla_{x,y} \widetilde{\omega}_e^n = F_e^n, \quad \widetilde{\omega}_e^n(0) = \widetilde{\omega}_{0,e}^n(x).$$

It is obvious that $\widetilde{\omega}_e^n = \widetilde{\omega}^n$ in $\overline{\mathcal{S}}$ by the uniqueness of the solution. By a standard $H^{s-\frac{1}{2}}$ energy estimate, we deduce that

$$\frac{d}{dt} \|\widetilde{\omega}_e^n(t)\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{d+1})} \leq C \|\overline{v}_e^n\|_{H^{s+\frac{1}{2}}(\mathbf{R}^{d+1})} \|\widetilde{\omega}_e^n(t)\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{d+1})} + \|F_e^n\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{d+1})},$$

from which and Gronwall's inequality, it follows that

$$\|\widetilde{\omega}_e^n(t)\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{d+1})} \leq (\|\widetilde{\omega}_{0,e}^n\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{d+1})} + \|F_e^n\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{d+1})}) e^{\int_0^t \|\overline{v}_e^n\|_{H^{s+\frac{1}{2}}(\mathbf{R}^{d+1})} dt'}.$$

This implies that

$$\|\widetilde{\omega}^n(t)\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} \leq C (\|\widetilde{\omega}_0^n\|_{H^{s-\frac{1}{2}}(\overline{\mathcal{S}})} + \mathcal{P}(E_s^n(t))) e^{\int_0^t \mathcal{P}(E_s^n(t')) dt'},$$

which together with (14.2) leads to

$$E_{s,l}^n(t) \leq E_{s,l}^n(0) + \int_0^t \mathcal{P}(E_s^n(t')) dt'. \quad (14.3)$$

Now let us turn to the higher order energy estimate. For this, we need the following refined elliptic estimate from [3].

Lemma 14.1. *Let $v \in H^{s+\frac{1}{2}}(\overline{\mathcal{S}})$ be a solution of the elliptic equation (4.11) with $v(0) = f$. Then it holds that for any $\sigma \in [-\frac{1}{2}, s-1]$,*

$$\|\nabla_{x,z} v\|_{X^\sigma([-\frac{1}{2}, 0])} \leq \mathcal{P}(\|\eta\|_{H^s}) (\|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma(I)}),$$

if it holds for $\sigma = -\frac{1}{2}$.

Lemma 14.2. *It holds that*

$$\begin{aligned} &\|U^n\|_{H^s} + \|(V^n, B^n)\|_{H^s} + \|\eta^n\|_{H^{s+\frac{1}{2}}} \\ &\leq \mathcal{P}(E_{s,l}^n) (\|D_t U^n\|_{H^{s-\frac{1}{2}}} + \|T_{\sqrt{a^n \lambda^n}} U^n\|_{H^{s-\frac{1}{2}}}) + \mathcal{P}(E_{s,l}^n). \end{aligned}$$

Proof. Step 1. Estimate for U^n .

Let $\epsilon \in (0, \min(s_d, 1))$ with $s_d = s - \frac{d}{2} - 1$. Applying Lemma 14.1 to P^n , we obtain

$$\|\nabla_{x,z} \widetilde{P}_1^n\|_{X^{s-1}([-\frac{1}{2}, 0])} \leq \mathcal{P}(E_{s,l}^n(t)), \quad P_1^n = P^n + y. \quad (14.4)$$

Then we deduce from Proposition 2.6 and (14.4) that

$$\begin{aligned} \|U^n\|_{H^s} &\leq \|T_{(\sqrt{a^n \lambda^n})^{-1}} T_{\sqrt{a^n \lambda^n}} U^n\|_{H^s} + \|(T_{(\sqrt{a^n \lambda^n})^{-1}} T_{\sqrt{a^n \lambda^n}} - 1) U^n\|_{H^s} \\ &\leq \mathcal{P}(E_{s,l}^n) \|T_{\sqrt{a^n \lambda^n}} U^n\|_{H^{s-\frac{1}{2}}} + \mathcal{P}(E_{s,l}^n) \|U^n\|_{H^{s-\epsilon}}, \end{aligned}$$

which implies that

$$\|U^n\|_{H^s} \leq \mathcal{P}(E_{s,l}^n) \|T_{\sqrt{a^n \lambda^n}} U^n\|_{H^s} + \mathcal{P}(E_{s,l}^n). \quad (14.5)$$

Step 2. Estimate for η^n

By the proof of Lemma 9.16, we know that

$$\zeta^n = T_{(a^n)^{-1}} (-D_t U^n + h_1^n + [D_t, T_{\zeta^n}] B^n) + (T_{(a^n)^{-1}} T_{a^n} - 1) \zeta^n,$$

which along with Proposition 2.6 and Proposition 2.21 gives

$$\begin{aligned} \|\zeta^n\|_{H^{s-\frac{1}{2}}} &\leq \mathcal{P}(E_{s,l}^n) (\|D_t U^n\|_{H^{s-\frac{1}{2}}} + \|\zeta^n\|_{H^{s-\frac{1}{2}-\epsilon}} + \|h_1^n\|_{H^{s-\frac{1}{2}}} \\ &\quad + \|V^n\|_{B_{\infty,1}^1} \|B^n\|_{H^{s-\frac{1}{2}}} + \|D_t \zeta^n\|_{L^\infty} \|B^n\|_{H^{s-\frac{1}{2}}}). \end{aligned}$$

Recall that $h_1^n = (T_{V^n} - V^n) \cdot \nabla V^n - R(a^n, \zeta^n) + T_{\zeta^n} (T_{V^n} - V^n) \cdot \nabla B^n$. Then Lemma 2.10 and (14.4) ensure that

$$\|h_1^n\|_{H^{s-\frac{1}{2}}} \leq \mathcal{P}(E_{s,l}^n) \|V^n\|_{W^{1,\infty}} \|(V^n, B^n)\|_{H^{s-\frac{1}{2}}} + \mathcal{P}(E_{s,l}^n) \|\zeta^n\|_{H^{s-\frac{1}{2}-\epsilon}}.$$

Recall that $(\partial_t + V^n \cdot \nabla) \zeta^n = \nabla B^n - \nabla V^n \cdot \nabla \eta^n$. Then we have

$$\|D_t \zeta^n(t)\|_{L^\infty} \leq C \|(V^n, B^n)\|_{W^{1,\infty}} \|\eta^n\|_{W^{1,\infty}}. \quad (14.6)$$

Thanks to $s > 1 + \frac{d}{2}$, we get by the interpolation that

$$\|(V^n, B^n)\|_{B_{\infty,1}^1} \leq \|(V^n, B^n)\|_{H^{s-\frac{1}{2}}}^{2\epsilon} \|(V^n, B^n)\|_{H^s}^{1-2\epsilon}. \quad (14.7)$$

Thus, we conclude that

$$\|\eta^n\|_{H^{s+\frac{1}{2}}} \leq \mathcal{P}(E_{s,l}^n(t)) + \mathcal{P}(E_{s,l}^n(t)) \|D_t U^n\|_{H^{s-\frac{1}{2}}} + \mathcal{P}(E_{s,l}^n(t)) \|(V^n, B^n)\|_{H^s}^{1-2\epsilon}. \quad (14.8)$$

Step 3. Estimates for B^n .

By the proof of Lemma 9.17, we know that

$$\operatorname{div} U^n = -T_{q^n} B^n - R(\eta^n) B^n + V_\omega^n + T_{\operatorname{div} \zeta^n} B^n,$$

where the symbol $q^n = \lambda^n - i\zeta^n \cdot \xi$ and

$$V_\omega^n = -\partial_y (v^n)_\omega^{d+1} + \nabla \eta^n \cdot \nabla (v^n)_\omega^{d+1} \big|_{y=\eta^n} + \partial_i \eta^n \omega_{d+1,i}^n \big|_{y=\eta^n}.$$

By using Lemma 14.1 and following the proof of Proposition 5.4, we can deduce that

$$\begin{aligned} \|R(\eta^n)(V^n, B^n)\|_{H^{s-1}} &\leq \mathcal{P}(E_{s,\ell}^n(t)) \|\eta^n\|_{H^{s+\frac{1}{2}}}, \\ \|\nabla_{x,z} \widetilde{v}_\omega^n\|_{X^{s-1}([-\frac{1}{2}, 0])} &\leq \mathcal{P}(E_{s,l}^n(t)). \end{aligned} \quad (14.9)$$

Thus, we infer from Lemma 2.10 that

$$\begin{aligned} \|T_{q^n} B^n\|_{H^{s-1}} &\leq \|U^n\|_{H^s} + \|R(\eta^n) B^n\|_{H^{s-1}} + \|V_\omega^n\|_{H^{s-1}} + \|\zeta^n\|_{C^\epsilon} \|B^n\|_{H^{s-\epsilon}} \\ &\leq \|U^n\|_{H^s} + \mathcal{P}(E_{s,l}^n) \|\eta^n\|_{H^{s+\frac{1}{2}}} + \mathcal{P}(E_{s,l}^n) \|B^n\|_{H^{s-\epsilon}}. \end{aligned} \quad (14.10)$$

Then Proposition 2.6 ensures that

$$\begin{aligned} \|B^n\|_{H^s} &\leq \|(T_{(q^n)^{-1}}T_{q^n} - 1)B^n\|_{H^s} + \|T_{(q^n)^{-1}}T_{q^n}B^n\|_{H^s} \\ &\leq \mathcal{P}(E_{s,l}^n)\|B^n\|_{H^{s-\epsilon}} + \mathcal{P}(E_{s,l}^n)\|T_{q^n}B^n\|_{H^{s-1}} \\ &\leq \mathcal{P}(E_{s,l}^n)(\|B^n\|_{H^{s-\epsilon}} + \|U^n\|_{H^s} + \mathcal{P}(E_{s,l}^n)\|\eta^n\|_{H^{s+\frac{1}{2}}}). \end{aligned} \quad (14.11)$$

Step 4. Completion of the estimate

Combining the estimates (14.12), (14.10) and (14.5), we have that

$$\begin{aligned} &\|U^n\|_{H^s} + \|B^n\|_{H^s} + \|\eta^n\|_{H^{s+\frac{1}{2}}} \\ &\leq \mathcal{P}(E_{s,l}^n)(\|D_t U^n\|_{H^{s-\frac{1}{2}}} + \|T_{\sqrt{a^n \lambda^n}} U^n\|_{H^{s-\frac{1}{2}}}) + \mathcal{P}(E_{s,l}^n). \end{aligned} \quad (14.12)$$

This also gives

$$\begin{aligned} \|V^n\|_{H^s} &\leq \|U^n\|_{H^s} + \|T_{\zeta^n} B^n\|_{H^s} \\ &\leq \mathcal{P}(E_{s,l}^n)(\|D_t U^n\|_{H^{s-\frac{1}{2}}} + \|T_{\sqrt{a^n \lambda^n}} U^n\|_{H^{s-\frac{1}{2}}}) + \mathcal{P}(E_{s,l}^n), \end{aligned}$$

which together with (14.5) and (14.12) gives the lemma. \square

Proceeding the same way as in section 9.6, we can deduce that

$$\begin{aligned} &\frac{d}{dt} \left(\|D_t U^n\|_{H^{s-\frac{1}{2}}}^2 + \|T_{\sqrt{a^n \lambda^n}} U^n\|_{H^{s-\frac{1}{2}}}^2 + \langle (f_\omega^n)_{s-1}, U_s^n \rangle \right. \\ &\quad \left. + \langle g_{s-1/2}^n, h_{s-1/2}^n \rangle + \frac{1}{2} \langle g_{s-1/2}^n, g_{s-1/2}^n \rangle \right) \leq \mathcal{P}(E_s^n). \end{aligned} \quad (14.13)$$

Here $g^n = [D_t, T_{\zeta^n}]B^n$ and $h^n = D_t U^n - [D_t, T_{\zeta^n}]B^n$.

It follows from Proposition 2.21 that

$$\begin{aligned} \|g^n\|_{H^{s-\frac{1}{2}}} &\leq C(\|V^n\|_{B_{\infty,1}^1} + \|D_t \zeta^n\|_{L^\infty})\|B^n\|_{H^{s-\frac{1}{2}}}, \\ \|h^n\|_{H^{s-\frac{1}{2}}} &\leq \|D_t U^n\|_{H^{s-\frac{1}{2}}} + C(\|V^n\|_{B_{\infty,1}^1} + \|D_t \zeta^n\|_{L^\infty})\|B^n\|_{H^{s-\frac{1}{2}}}, \end{aligned}$$

and we have by (14.9) that

$$\|f_\omega^n\|_{H^{s-1}} \leq \mathcal{P}(E_{s,l}^n(t)).$$

Plugging the above estimates into (14.13), we obtain

$$\begin{aligned} &\|D_t U^n(t)\|_{H^{s-\frac{1}{2}}}^2 + \|T_{\sqrt{a^n \lambda^n}} U^n(t)\|_{H^{s-\frac{1}{2}}}^2 \\ &\leq \mathcal{P}(E_s(0)) + \mathcal{P}(E_{s,l}^n(t))\|U^n(t)\|_{H^s} + C(\|V^n(t)\|_{B_{\infty,1}^1} + \|D_t \zeta^n(t)\|_{L^\infty})^2 \|B^n(t)\|_{H^{s-\frac{1}{2}}}^2 \\ &\quad + C(\|V^n(t)\|_{B_{\infty,1}^1} + \|D_t \zeta^n(t)\|_{L^\infty})\|B^n(t)\|_{H^{s-\frac{1}{2}}} \|D_t U^n\|_{H^{s-\frac{1}{2}}} + \int_0^t \mathcal{P}(E_s^n(t')) dt', \end{aligned}$$

which together with (14.6), (14.7) and Lemma 14.2 implies that

$$E_s^n(t) \leq \mathcal{P}(E_s(0)) + \mathcal{P}(E_{s,l}^n(t)) + \int_0^t \mathcal{P}(E_s^n(t')) dt'.$$

This together with (14.3) ensures that there exists $T > 0$ depending on $E_s(0)$ and c_0, h_0, s such that for any $t \in [0, \min(T, T_n))$, there holds

$$E_s^n(t) \leq \mathcal{P}(E_s(0)).$$

With the uniform estimates, the existence of the solution can be deduced by a standard compact argument. Here we omit the details.

14.3. Uniqueness of the solution. Let (η^1, v^1, P^1) and (η^2, v^2, P^2) be two solutions of the system (1.1)–(1.5) with the same initial data. Assume that the solutions satisfy

$$\sup_{t \in [0, T]} (\|\eta^1(t)\|_{H^{s+\frac{1}{2}}} + \|\tilde{v}^1(t)\|_{H^{s+\frac{1}{2}}(\overline{S})} + \|\eta^2(t)\|_{H^{s+\frac{1}{2}}} + \|\tilde{v}^2(t)\|_{H^{s+\frac{1}{2}}(\overline{S})}) \leq \mathcal{P}_0,$$

Here and in what follows, we denote by \mathcal{P}_0 a constant depending only on $E_s(0), c_0, h_0, s$, which may change from line to line.

We denote $\delta_f = f^2 - f^1$ and introduce

$$\delta_{\mathcal{E}}(t) \stackrel{\text{def}}{=} \delta_{\mathcal{E}_1}(t) + \delta_{\mathcal{E}_2}(t),$$

where $\delta_{\mathcal{E}_1}(t)$ and $\delta_{\mathcal{E}_2}(t)$ are given by

$$\delta_{\mathcal{E}_1}(t) = \|(\delta_V, \delta_B)(t)\|_{H^{s-1}} + \|\delta_{\eta}(t)\|_{H^{s-\frac{1}{2}}},$$

$$\delta_{\mathcal{E}_2}(t) = \|(\delta_V, \delta_B)(t)\|_{H^{s-\frac{3}{2}}} + \|(\delta_{V_b}, \delta_{\eta})(t)\|_{H^{s-1}} + \|\delta_{\tilde{\omega}}(t)\|_{H^{s-\frac{3}{2}}(\overline{S})}.$$

In what follows, we denote by $L_i (i = 0, 1, 2)$ some nonlinear term, which satisfies

$$\|L_i(t)\|_{H^{s-1-\frac{i}{2}}} \leq \mathcal{P}_0 \delta_{\mathcal{E}}(t).$$

We denote $D_t = \partial_t + V^2 \cdot \nabla$.

Following the proof of section 12.2, we can show that

$$\|D_t \delta_{R_{\omega}}\|_{H^{s-2}} + \|(\delta_{R(\eta)(V, B)}, \delta_a)\|_{H^{s-\frac{3}{2}}} + \|\delta_{\nabla P|_{y=-1}}\|_{H^{s-1}} \leq \mathcal{P}_0 \delta_{\mathcal{E}}(t), \quad (14.14)$$

$$\|\delta_{R_{\omega}}\|_{H^{s-2}} \leq \mathcal{P}_0 \delta_{\mathcal{E}_2}(t). \quad (14.15)$$

With (14.14) and (14.15), we can deduce that

$$\begin{cases} D_t(\delta_V, \delta_B) = L_1, \\ (\partial_t + V_b^2 \cdot \nabla) \delta_{V_b} = L_0, \\ D_t \delta_{\zeta} = L_2, \\ D_t \delta_U + T_{a^2 \lambda^2} \delta_U = L_1 + T_{a^2} \delta_{R_{\omega}}. \end{cases}$$

The energy estimate ensures that

$$\begin{aligned} & \|D_t \delta_U(t)\|_{H^{s-\frac{3}{2}}} + \|T_{\sqrt{a^2 \lambda^2}} \delta_U(t)\|_{H^{s-\frac{3}{2}}} + \|(\delta_V, \delta_B)(t)\|_{H^{s-\frac{3}{2}}} \\ & + \|(\delta_{V_b}, \delta_{\eta})(t)\|_{H^{s-1}} \leq \mathcal{P}_0 \int_0^t \delta_{\mathcal{E}}(t') dt'. \end{aligned}$$

For the vorticity, we have

$$\|\delta_{\tilde{\omega}}(t)\|_{H^{s-\frac{3}{2}}(\overline{S})} \leq \mathcal{P}_0 \int_0^t \delta_{\mathcal{E}}(t') dt'.$$

Using the fact that

$$\delta_{\mathcal{E}}(t) \leq \mathcal{P}_0 (\delta_{\mathcal{E}_2}(t) + \|D_t \delta_U(t)\|_{H^{s-\frac{3}{2}}} + \|T_{\sqrt{a^2 \lambda^2}} \delta_U(t)\|_{H^{s-\frac{3}{2}}}).$$

We conclude that

$$\delta_{\mathcal{E}}(t) \leq \mathcal{P}_0 \int_0^t \delta_{\mathcal{E}}(t') dt',$$

which implies that $\delta_{\mathcal{E}}(t) = 0$ for any $t \in [0, T]$. The uniqueness is proved. \square

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